Computing with SAT Oracles

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What is SAT?

- SAT is the decision problem for propositional logic
  - Well-formed propositional formulas, with variables, logical connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and parenthesis: $($, $)$
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- SAT is NP-complete [Coo71]

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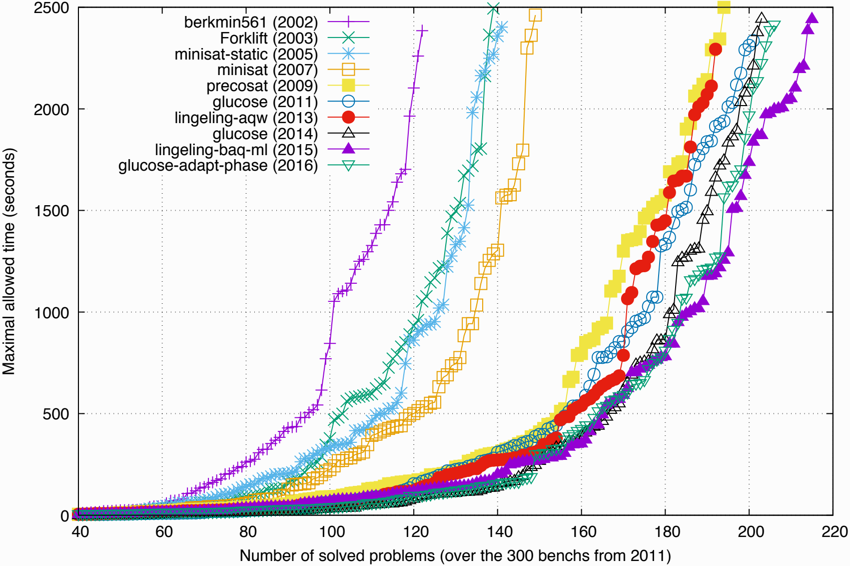
- CDCL SAT solving is a success story of Computer Science
  - Conflict-Driven Clause Learning (CDCL)
  - (CDCL) SAT has impacted many different fields
  - Hundreds (thousands?) of practical applications

CDCL SAT solver improvement

[Source: Simon 2015]



How good are SAT solvers?

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- Sample SAT of solvers:
  1. POSIT: state of the art DPLL SAT solver in 1995
  2. GRASP: first CDCL SAT solver, state of the art 1995$\sim$2000
  3. Minisat: CDCL SAT solver, state of the art until the late 00s
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- Example 1: model checking example (from IBM)
- Example 2: cooperative path finding (CPF)

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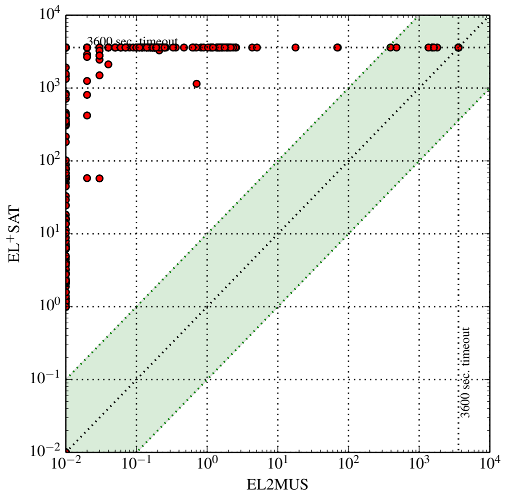
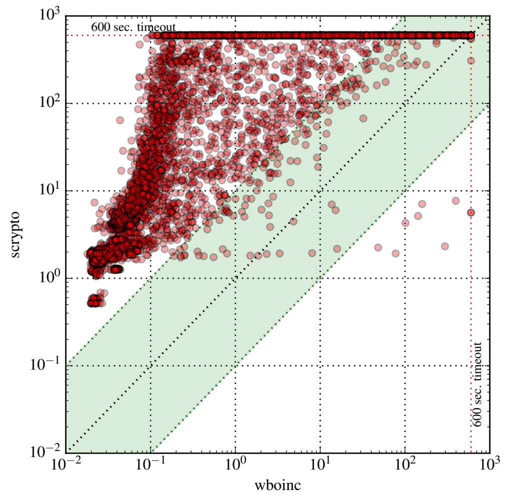
- Search space with 2832875 propositional variables (worst case):
  - # of assignments to $> 2.8 \times 10^{6}$ variables: $\gg 10^{840000}$ **!!**
  - **Obs:** SAT solvers at present (but formula dependent)

SAT can make the difference – axiom pinpointing



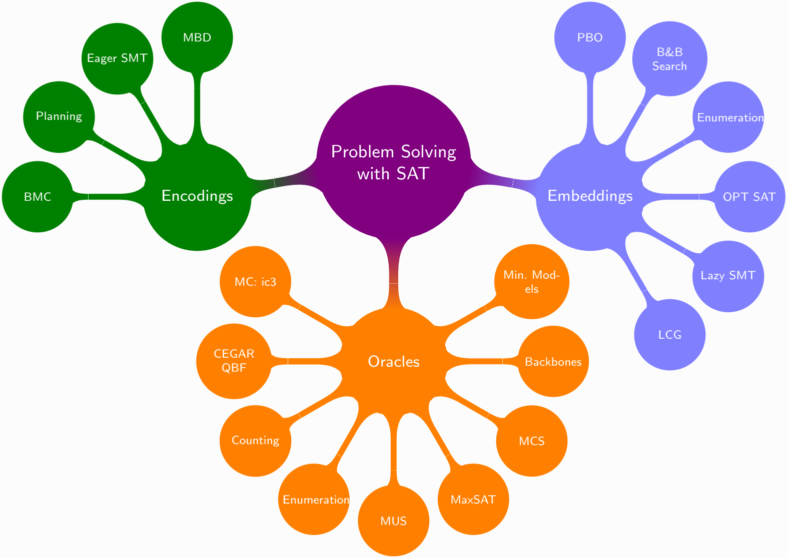
- $\mathcal{EL}^+$ medical ontologies [AMM15]
  - Minimal unsatisfiability (MUSes) & maximal satisfiability (MCSes) & Enumeration

SAT can make the difference – model based diagnosis



- Model-based diagnosis problem instances [MJIM15]
  - Maximum satisfiability (MaxSAT)

CDCL SAT is ubiquitous in problem solving



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  - Quantification problems; Counting problems; Etc.

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Part 0

Basic Definitions

Preliminaries

- Variables: $w, x, y, z, a, b, c, \ldots$
- Literals: $w, \bar{x}, \bar{y}, a, \ldots$, but also $\neg w, \neg y, \ldots$
- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses
- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$ that satisfies formula
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- Example:

$$\mathcal{F} \triangleq (r) \wedge (\bar{r} \vee s) \wedge (\overline{w} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)$$

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$$\mathcal{F} \triangleq (r) \land (\bar{r} \lor s) \land (\overline{w} \lor a) \land (\bar{x} \lor b) \land (\bar{y} \lor \bar{z} \lor c) \land (\bar{b} \lor \bar{c} \lor d)$$

  - Example models:
    - $\{r, s, a, b, c, d\}$
    - $\{r, s, \bar{x}, y, \bar{w}, z, \bar{a}, b, c, d\}$

Resolution

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$$\begin{tikzcd}
(x \vee a) \arrow[d, dash] & & (\bar{x} \vee a) \arrow[dll, dash] & & (\bar{y} \vee \bar{a}) \arrow[drr, dash] & & (y \vee \bar{a}) \arrow[d, dash] \\
(a) \arrow[drrr, dash] & & & & & & (\bar{a}) \arrow[dlll, dash] \\
& & & \perp & & &
\end{tikzcd}$$

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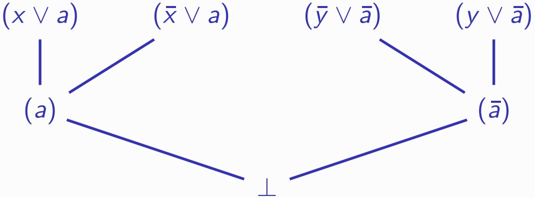

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- Self-subsuming resolution (with $\alpha' \subseteq \alpha$): [SP04, SB09]

$$\frac{(\alpha \vee x) \qquad\qquad (\alpha' \vee \bar{x})}{(\alpha)}$$

  - (α) subsumes $(\alpha \vee x)$

Unit propagation

$$\begin{aligned}\mathcal{F} \quad = \quad & (r) \wedge (\bar{r} \vee s) \wedge \\ & (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b) \wedge \\ & (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)\end{aligned}$$

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  $w = 1, x = 1, y = 1, z = 1$

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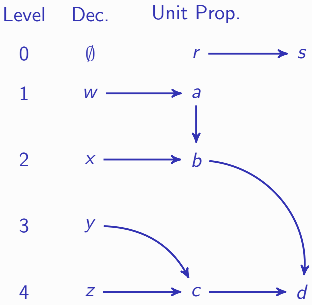

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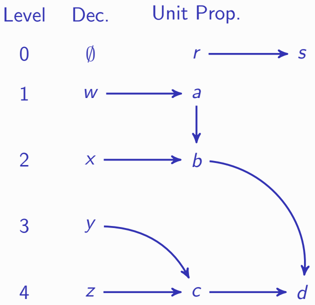

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- Additional definitions:
  - Antecedent (or reason) of an implied assignment
    - $(\bar{b} \vee \bar{c} \vee d)$ for d
  - Associate assignment with decision levels
    - $w = 1 \,@\, 1$, $x = 1 \,@\, 2$, $y = 1 \,@\, 3$, $z = 1 \,@\, 4$
    - $r = 1 \,@\, 0$, $d = 1 \,@\, 4$, ...

Resolution proofs

- Refutation of unsatisfiable formula by iterated resolution operations produces resolution proof
- An example:
  $\mathcal{F} = (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$
- Resolution proof:

$$
\begin{tikzcd}[column sep=small]
 & & (a \vee b) \arrow[dr, no head] & & (\bar{a} \vee c) \arrow[dl, no head] \\
 & (\bar{c}) \arrow[dr, no head] & & (b \vee c) \arrow[dl, no head] & \\
(\bar{b}) \arrow[dr, no head] & & (b) \arrow[dl, no head] & & \\
 & \perp & & &
\end{tikzcd}
$$

- A modern SAT solver can generate resolution proofs using clauses learned by the solver [ZM03]

- CNF formula:

$$\mathcal{F} \;=\; (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$$

Level	Dec.	Unit Prop.
0	$\emptyset$	$\begin{tikzcd} \bar{b} \arrow[r] & a \arrow[d] \\ \bar{c} \arrow[r] & \perp \end{tikzcd}$

Implication graph with conflict

Unsatisfiable cores & proof traces

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Proof trace $\perp$: $(\bar{a} \vee c)$ $(a \vee b)$ $(\bar{c})$ $(\bar{b})$

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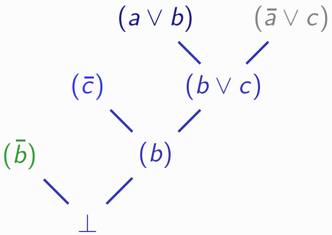

Resolution proof follows structure of conflicts

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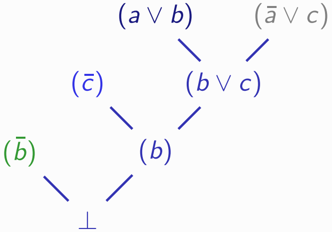
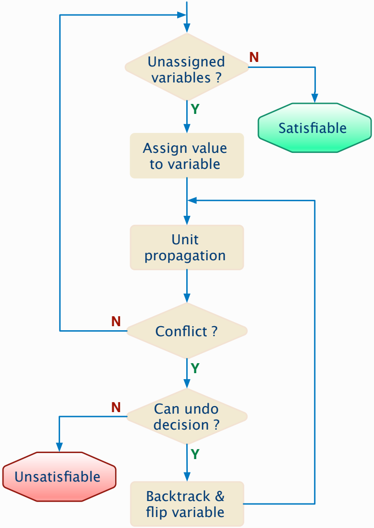


Unsatisfiable subformula (core): $(\bar{c}), (\bar{b}), (\bar{a} \vee c), (a \vee b)$

The DPLL algorithm

[DP60, DLL62]



- Optional: pure literal rule

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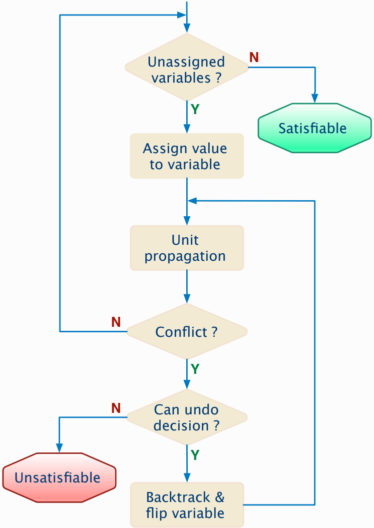

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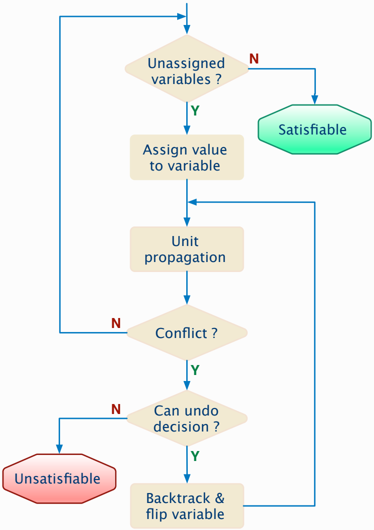

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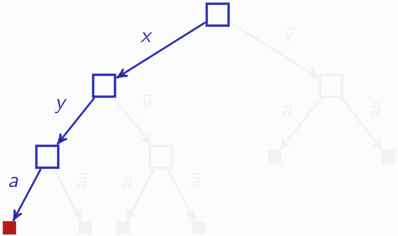

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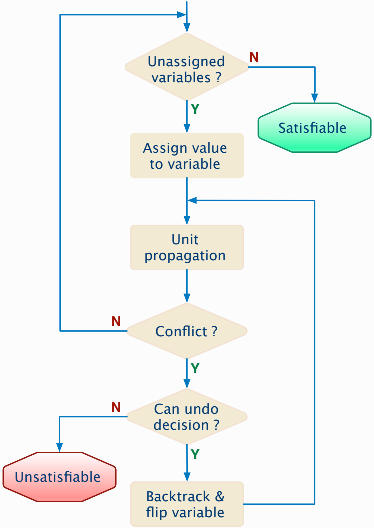

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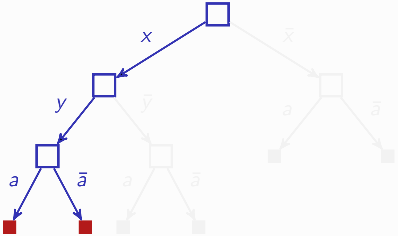

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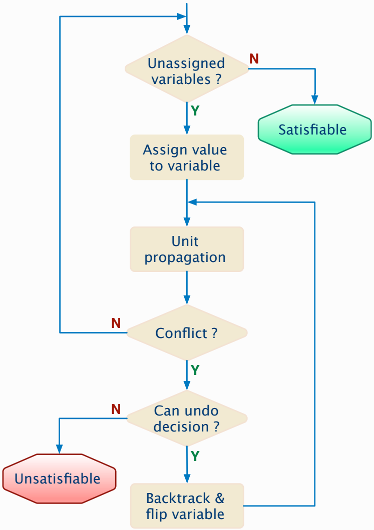

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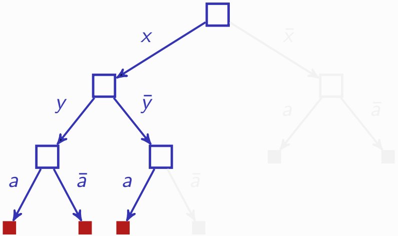

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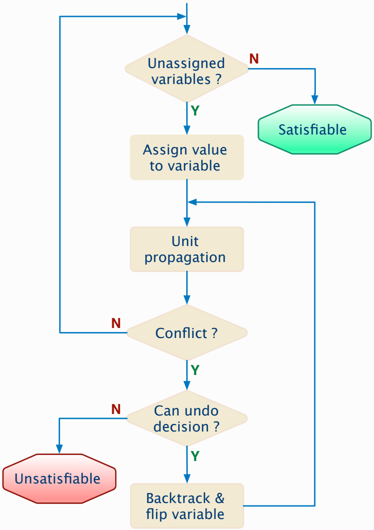

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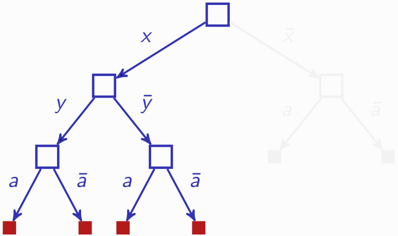

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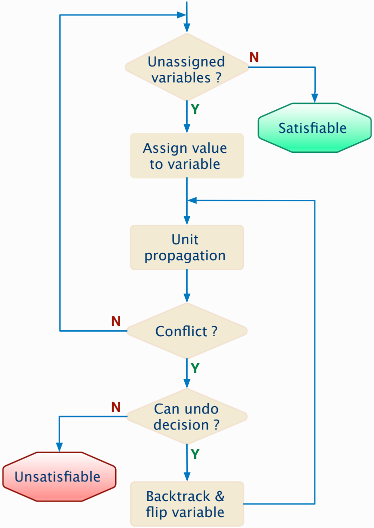

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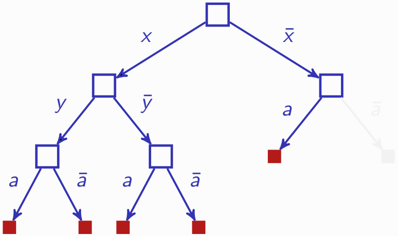

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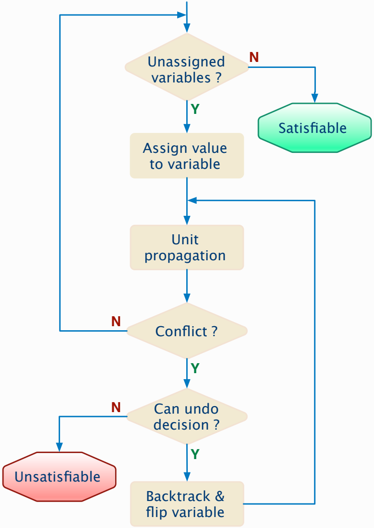

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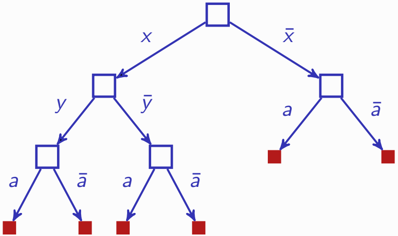
2	$\bar{a}$	$\longrightarrow \bar{b} \longrightarrow \perp$



- Optional: pure literal rule

Part 1

CDCL SAT Solving

What is a CDCL SAT solver?

- Extend DPLL SAT solver with: [DP60, DLL62]
  - Clause learning & non-chronological backtracking [MS95, MSS96b, MSS99]
  - Search restarts [GSC97, BMS00, Hua07, Bie08, LOM+18]
  - Lazy data structures
  - Conflict-guided branching
  - ...

What is a CDCL SAT solver?

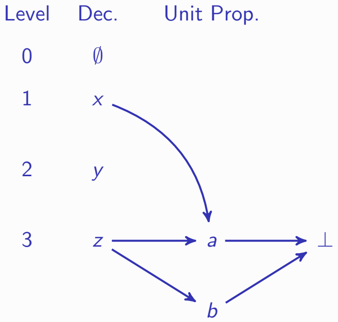
- Extend DPLL SAT solver with: [DP60, DLL62]
  - Clause learning & non-chronological backtracking [MS95, MSS96b, MSS99]
    - Exploit UIPs [MS95, MSS99, ZMMM01, SSS12]
    - Minimize learned clauses [SB09, Gel09, LLX$^+$17]
    - Opportunistically delete clauses [MSS96b, MSS99, GN02, AS09]
  - Search restarts [GSC97, BMS00, Hua07, Bie08, LOM$^+$18]
  - Lazy data structures
    - Watched literals [MMZ$^+$01]
  - Conflict-guided branching
    - Lightweight branching heuristics [MMZ$^+$01]
    - Phase saving [PD07]
  - ...

Outline

Clause learning

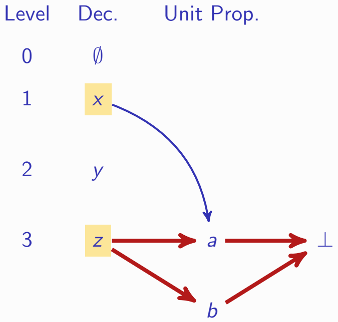
$$
\begin{tikzcd}
\text{Level} & \text{Dec.} & \text{Unit Prop.} & \\
0 & \emptyset & & \\
1 & x \arrow[ddr, bend left] & & \\
2 & y & & \\
3 & z \arrow[r] \arrow[dr] & a \arrow[r] & \perp \\
 & & b \arrow[ur] &
\end{tikzcd}
$$

Clause learning



- Analyze conflict [MS95, MSS96a, MSS96a, MSS96b, MSS99]

Clause learning



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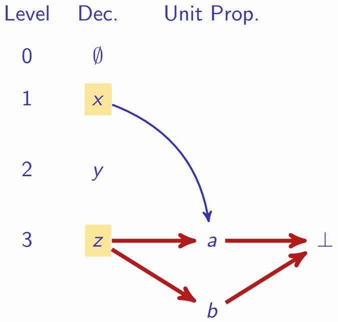
  - Reasons: x and z
    - Decision variable & literals assigned at decision levels less than current

Clause learning



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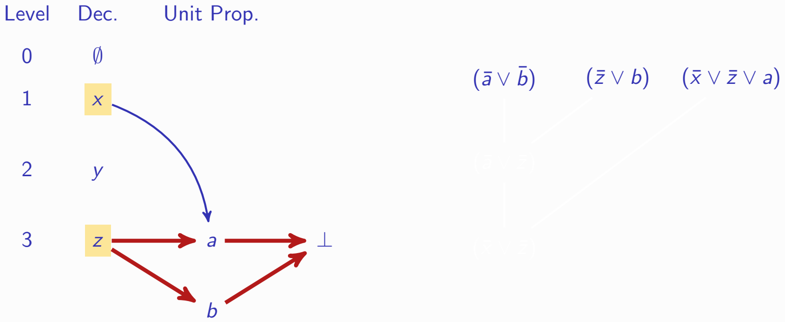
Clause learning



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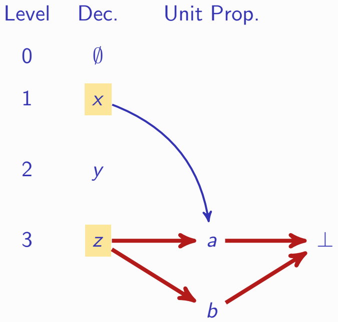
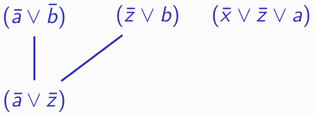
Clause learning





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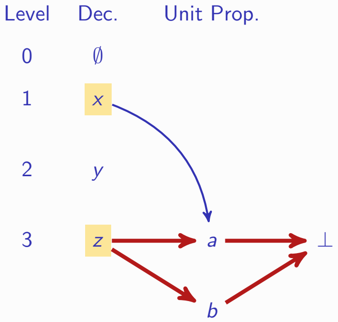

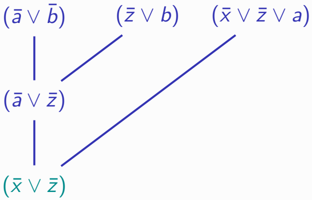

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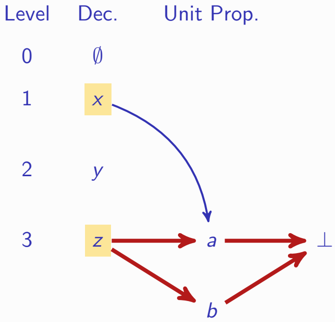
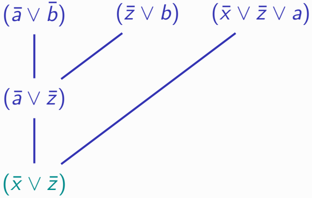
Clause learning





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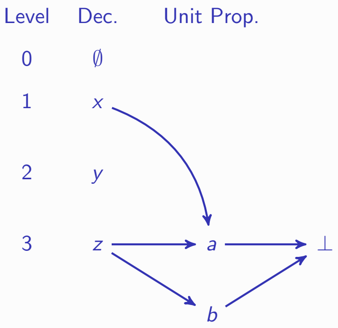
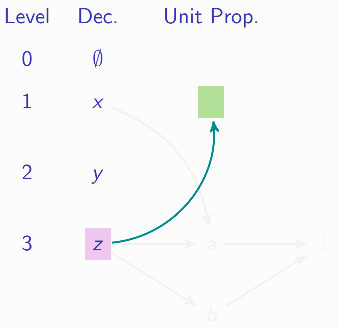
  - Reasons: x and z
    - Decision variable & literals assigned at decision levels less than current
  - Create **new** clause: $(\bar{x} \vee \bar{z})$
- Can relate clause learning with resolution
  - Learned clauses result from (**selected**) resolution operations

Clause learning – after backtracking

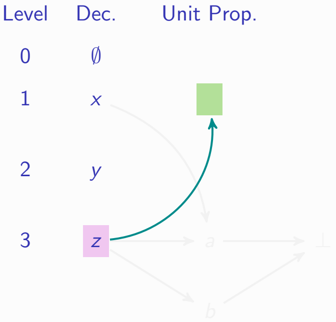


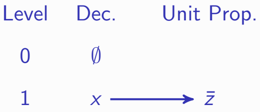
Clause learning – after backtracking



- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1

Clause learning – after backtracking





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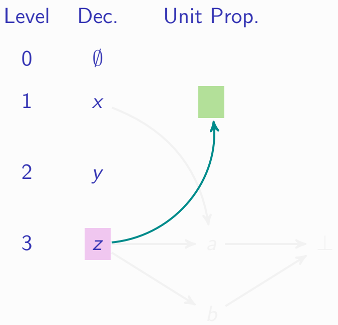
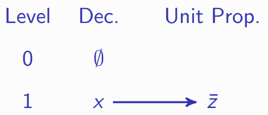
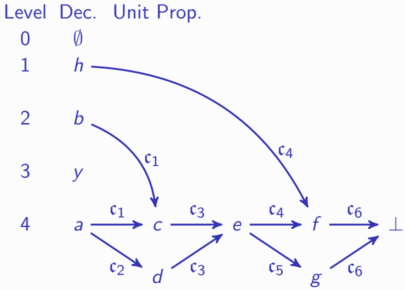


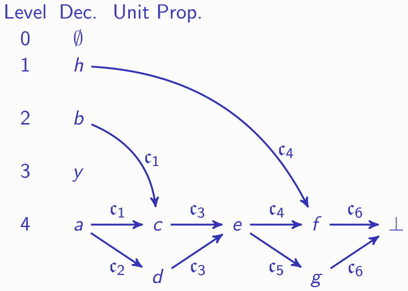


- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1
- Learned clauses are asserting (with exceptions) [MS95, MSS96b, MSS99]
- Backtracking differs from plain DPLL:
  - Always bactrack after a conflict [MMZ+01]

Quiz – conflict analysis



Quiz – conflict analysis



Step	Var Queue	Extract Var	Antecedent	Recorded Lits	Added to Queue
0	–	$\perp$	$\mathfrak{c}_6$	$\emptyset$	$\{f, g\}$

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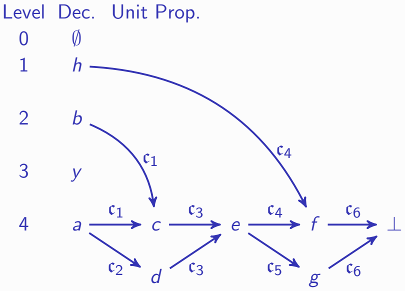

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0	–	$\perp$	$\mathfrak{c}_6$	$\emptyset$	$\{f, g\}$
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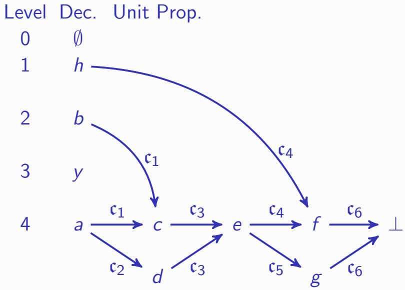

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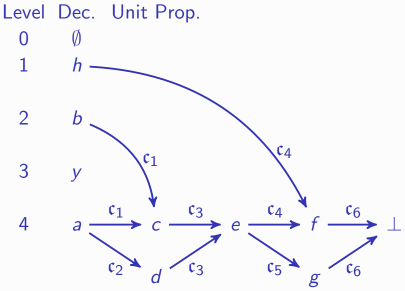

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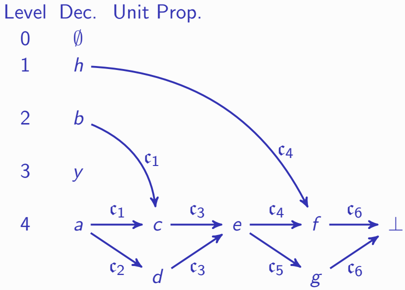
2	$[g, e]$	g	$\mathfrak{c}_5$	$\{\bar{h}\}$	$\emptyset$
3	$[e]$	e	$\mathfrak{c}_3$	$\{\bar{h}\}$	$\{c, d\}$

Quiz – conflict analysis



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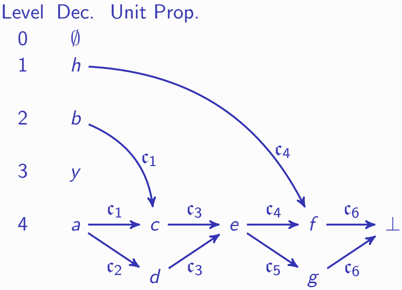
4	$[c, d]$	c	$\mathfrak{c}_1$	$\{\bar{h}, \bar{b}\}$	$\{a\}$

Quiz – conflict analysis



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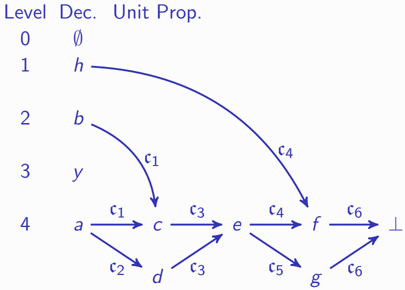
5	$[d, a]$	d	$\mathfrak{c}_2$	$\{\bar{h}, \bar{b}\}$	$\emptyset$

Quiz – conflict analysis



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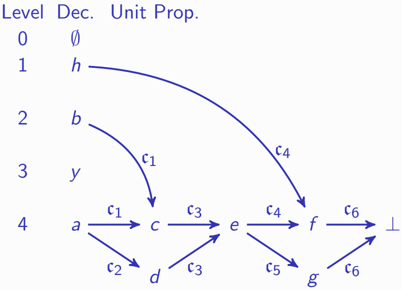
6	$[a]$	a	dec var	$\{\bar{h}, \bar{b}, \bar{a}\}$	–

Quiz – conflict analysis



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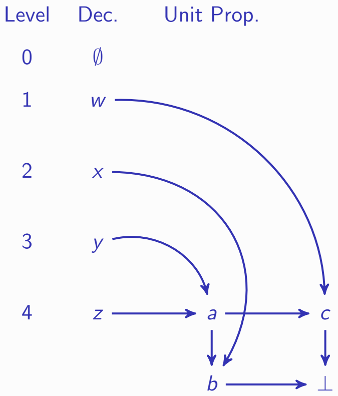
7	$[]$	–	–	$\{\bar{h}, \bar{b}, \bar{a}\}$	–

Unique implication points (UIPs)



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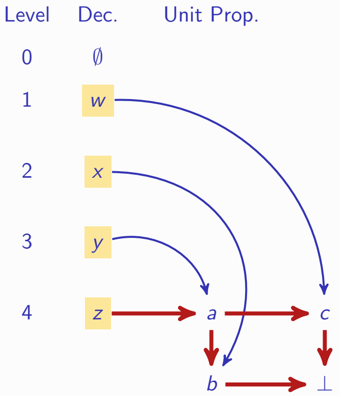
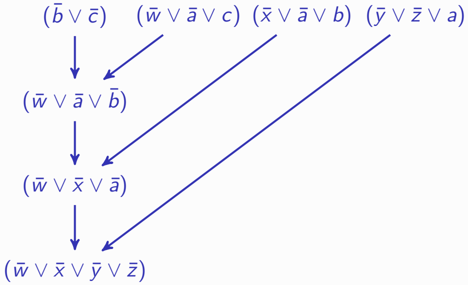
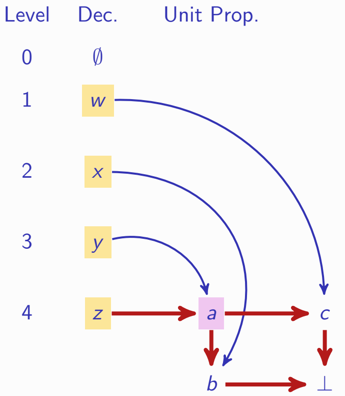
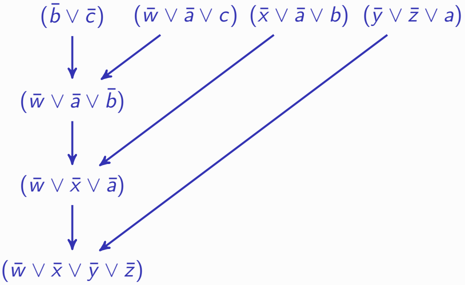




- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$





- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$
- But a is an UIP
  - Dominator in DAG for decision level 4

[MS95, MSS99]

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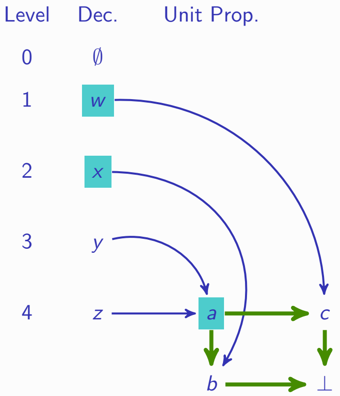

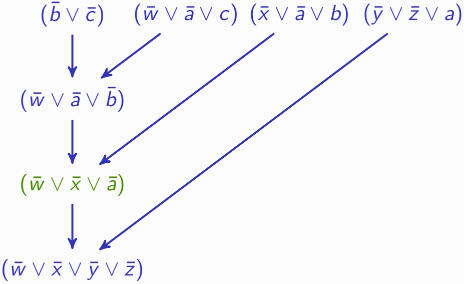

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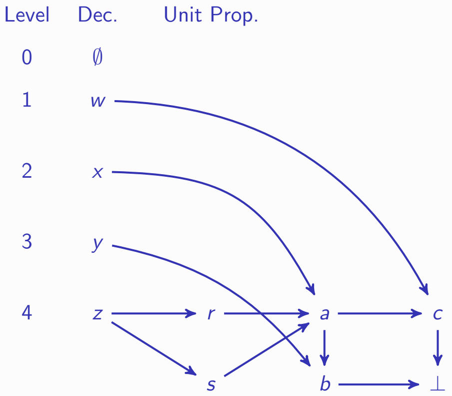
- But a is an UIP
  - Dominator in DAG for level 4
- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{a})$

[MS95, MSS99]

Multiple UIPs



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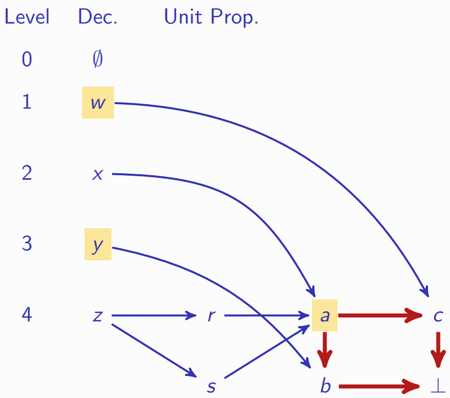

- First UIP:
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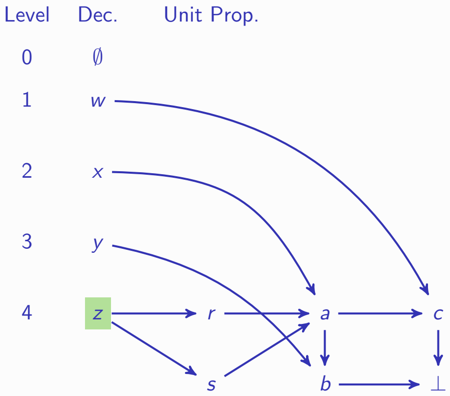
Multiple UIPs



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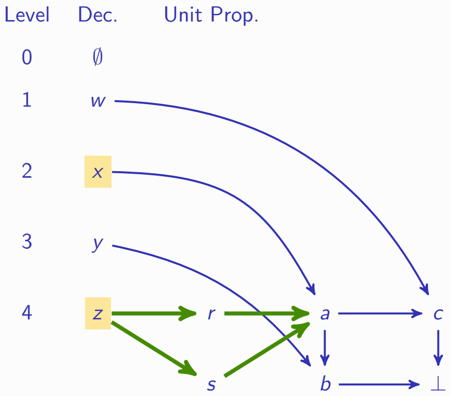

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- Second UIP:
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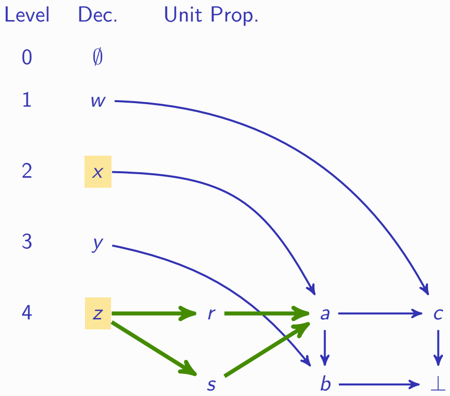
  - Clause is not asserting

Multiple UIPs



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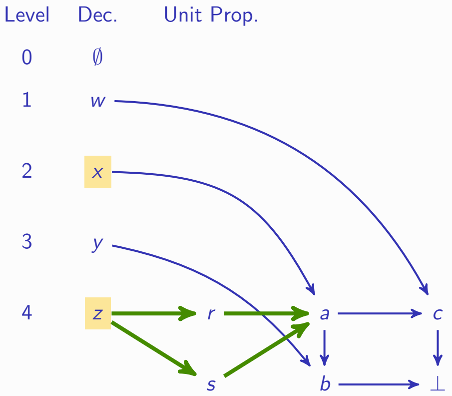
- In practice smaller clauses more effective
  - Compare with $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

Multiple UIPs



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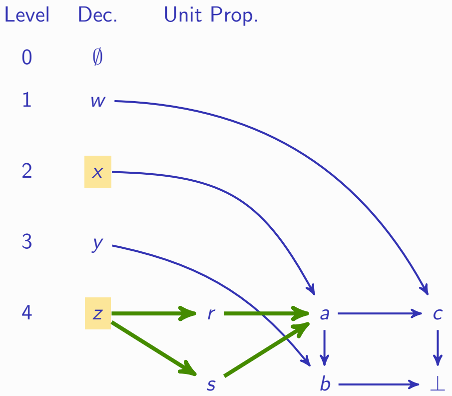
- Multiple UIPs proposed in GRASP [MS95, MSS99]
  - First UIP learning proposed in Chaff [MMZ+01]
- Not used in recent state of the art CDCL SAT solvers

Multiple UIPs



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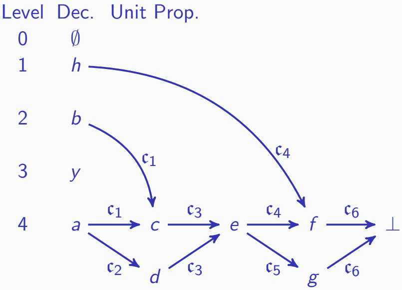
- Multiple UIPs proposed in GRASP [MS95, MSS99]
  - First UIP learning proposed in Chaff [MMZ+01]
- Not used in recent state of the art CDCL SAT solvers
- Recent results show it can be beneficial on some instances [SSS12]

Quiz – conflict analysis with UIP(s)



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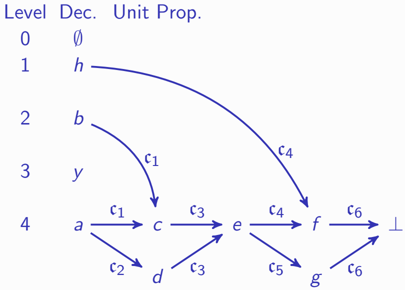

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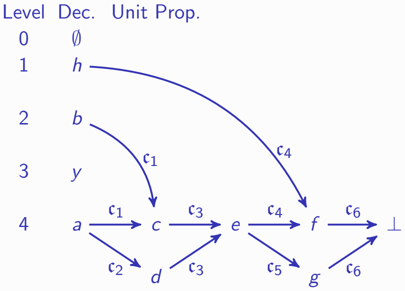

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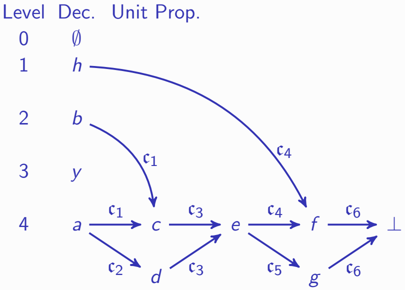
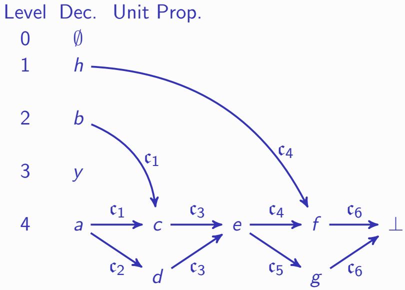
1	$[f, g]$	f	$\mathfrak{c}_4$	$\{\bar{h}\}$	$\{e\}$

Quiz – conflict analysis with UIP(s)



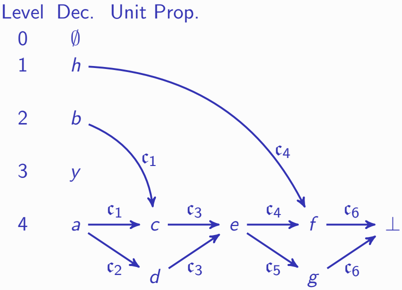
Step	Var Queue	Extract Var	Antecedent	Recorded Lits	Added to Queue
0	–	$\perp$	$\mathfrak{c}_6$	$\emptyset$	$\{f, g\}$
1	$[f, g]$	f	$\mathfrak{c}_4$	$\{\bar{h}\}$	$\{e\}$
2	$[g, e]$	g	$\mathfrak{c}_5$	$\{\bar{h}\}$	$\emptyset$

Quiz – conflict analysis with UIP(s)



Step	Var Queue	Extract Var	Antecedent	Recorded Lits	Added to Queue
0	–	$\perp$	$\mathfrak{c}_6$	$\emptyset$	$\{f, g\}$
1	$[f, g]$	f	$\mathfrak{c}_4$	$\{\bar{h}\}$	$\{e\}$
2	$[g, e]$	g	$\mathfrak{c}_5$	$\{\bar{h}\}$	$\emptyset$
3	$[e]$	e	$\mathfrak{c}_3$	$\{\bar{h}, \bar{e}\}$	$\emptyset$

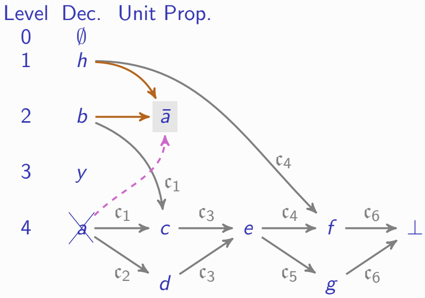
Quiz – conflict analysis with UIP(s)



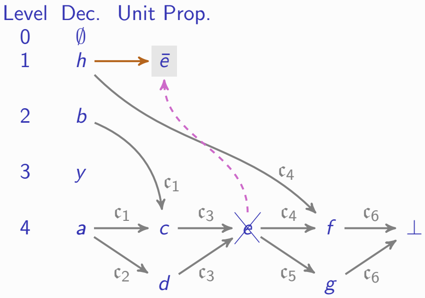
Step	Var Queue	Extract Var	Antecedent	Recorded Lits	Added to Queue
0	–	$\perp$	$\mathfrak{c}_6$	$\emptyset$	$\{f, g\}$
1	$[f, g]$	f	$\mathfrak{c}_4$	$\{\bar{h}\}$	$\{e\}$
2	$[g, e]$	g	$\mathfrak{c}_5$	$\{\bar{h}\}$	$\emptyset$
3	$[e]$	e	$\mathfrak{c}_3$	$\{\bar{h}, \bar{e}\}$	$\emptyset$
6	$[]$	–	–	$\{\bar{h}, \bar{e}\}$	–

Quiz (Cont.) – non-chronological backtracking

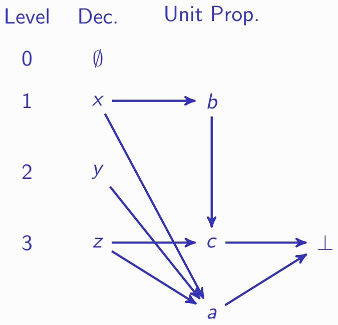
Without UIP:



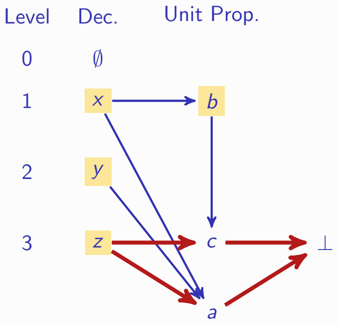
With UIP:

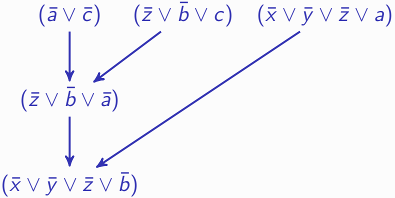


Clause minimization I



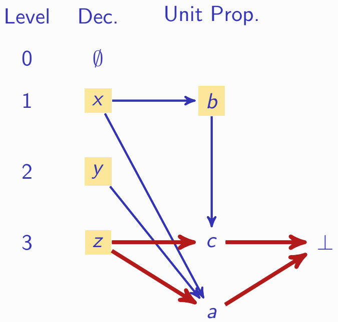
Clause minimization I

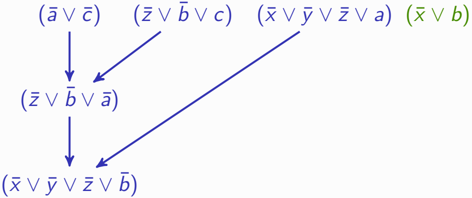




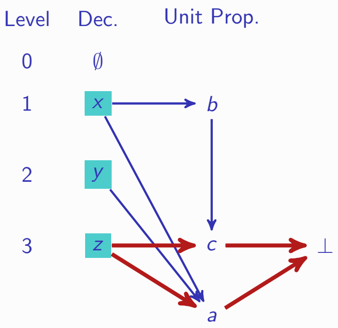
- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$

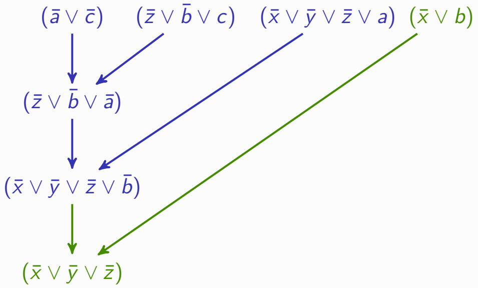
Clause minimization I





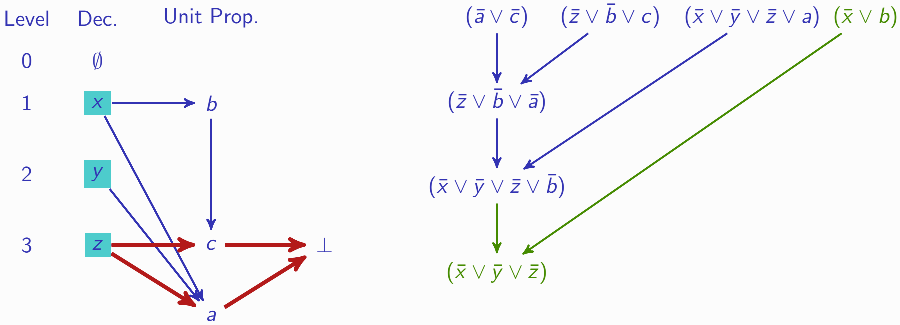
- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$
- Apply self-subsuming resolution (i.e. local minimization) [SB09]





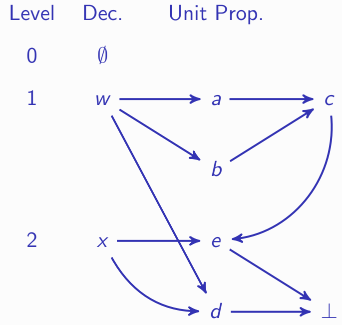
- ~~Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$~~
- Apply self-subsuming resolution (i.e. local minimization) [SB09]

Clause minimization I

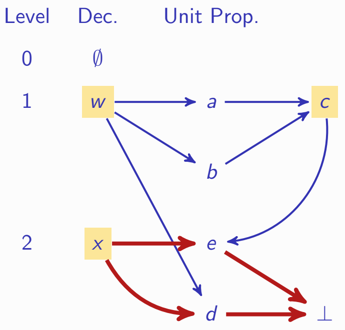


- ~~Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$~~
- Apply self-subsuming resolution (i.e. local minimization) [SB09]
- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z})$

Clause minimization II

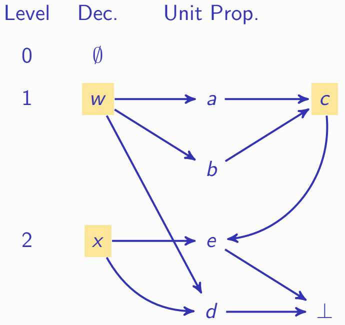


Clause minimization II



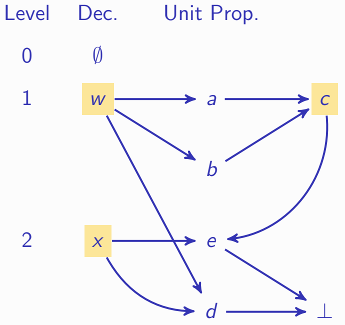
- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$

Clause minimization II



- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$

Clause minimization II



- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$
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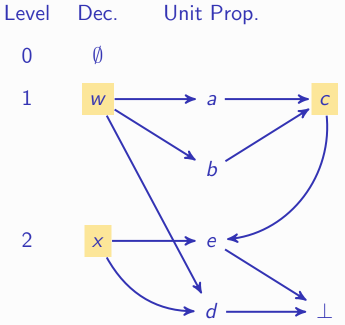
- Can apply recursive minimization

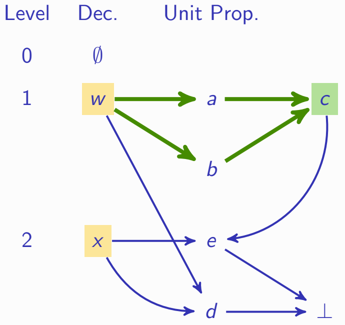
Clause minimization II



- ~~Learn clause ($\bar{w} \vee \bar{x} \vee \bar{c}$)~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields ($\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b}$)
- Can apply recursive minimization

- Marked nodes: literals in learned clause [SB09]

Clause minimization II



- ~~Learn clause ($\bar{w} \vee \bar{x} \vee \bar{c}$)~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields ($\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b}$)
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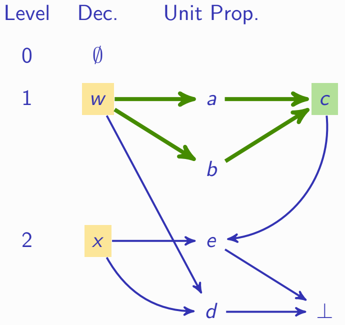
- Marked nodes: literals in learned clause [SB09]
- Trace back from c until marked nodes or new decision nodes
  - Drop literal c if only marked nodes visited

Clause minimization II



- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$
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- Learn clause $(\bar{w} \vee \bar{x})$

- Marked nodes: literals in learned clause [SB09]
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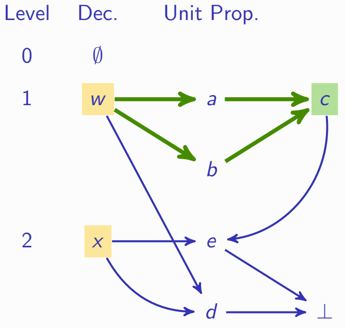
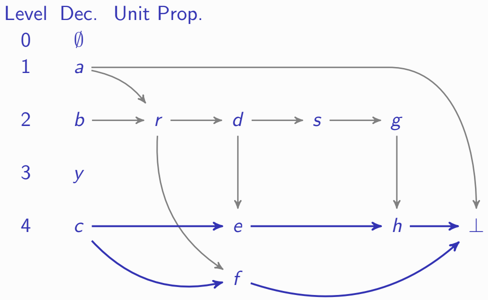
Clause minimization II



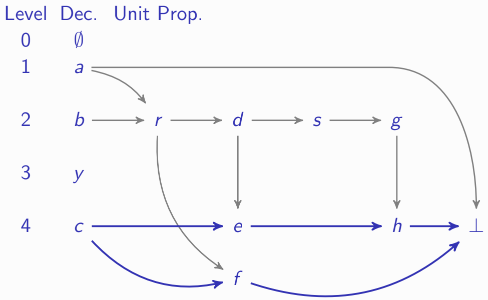
- ~~Learn clause ($\bar{w} \vee \bar{x} \vee \bar{c}$)~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields ($\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b}$)
- Can apply recursive minimization
- Learn clause ($\bar{w} \vee \bar{x}$)

- Marked nodes: literals in learned clause [SB09]
- Trace back from c until marked nodes or new decision nodes
  - Drop literal c if only marked nodes visited
- Recursive minimization runs in (amortized) linear time

Quiz – conflict clause minimization



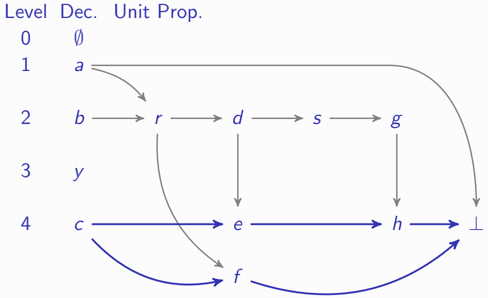
Quiz – conflict clause minimization



Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Quiz – conflict clause minimization

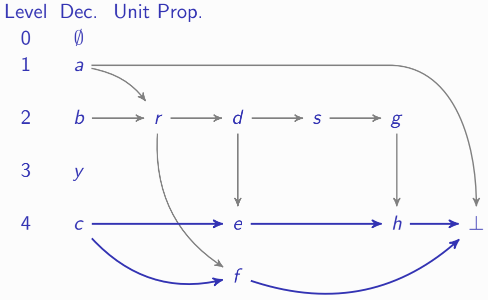


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action

Quiz – conflict clause minimization

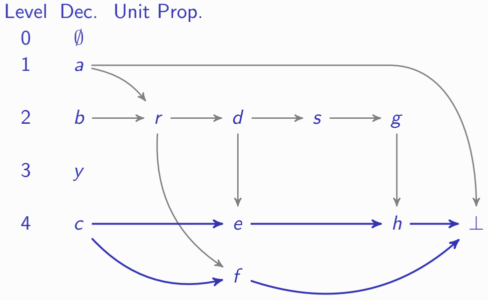


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–

Quiz – conflict clause minimization

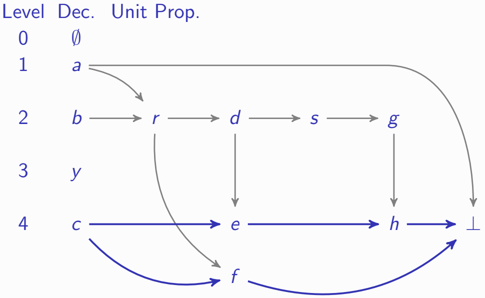


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–

Quiz – conflict clause minimization

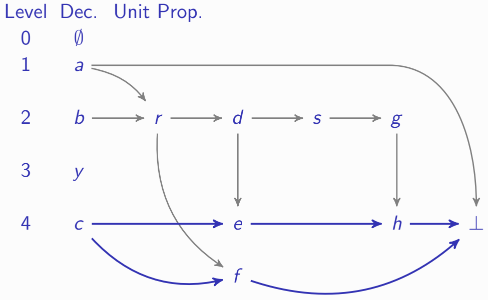


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip

Quiz – conflict clause minimization

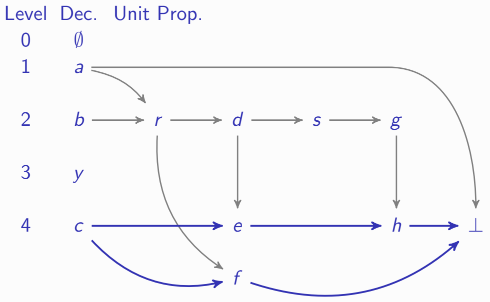


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \quad)$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g

Quiz – conflict clause minimization

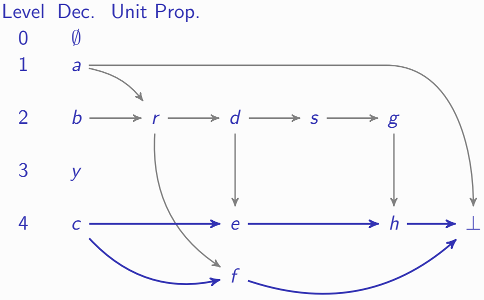


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \quad)$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g
d	d	$\{a, r, c\}$	$\emptyset$	$[r]$	–

Quiz – conflict clause minimization

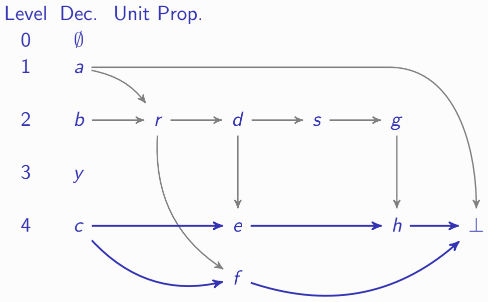


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \quad)$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g
d	d	$\{a, r, c\}$	$\emptyset$	$[r]$	–
d	r	$\{a, r, c\}$	$\emptyset$	$[]$	r marked, skip

Quiz – conflict clause minimization

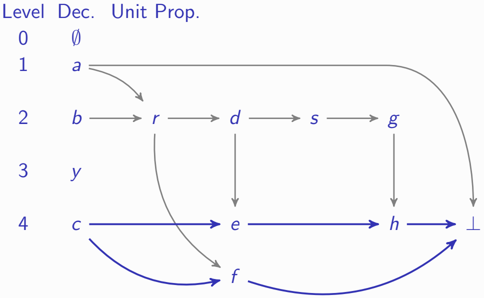


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g
d	d	$\{a, r, c\}$	$\emptyset$	$[r]$	–
d	r	$\{a, r, c\}$	$\emptyset$	$[]$	r marked, skip
d	–	$\{a, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop d

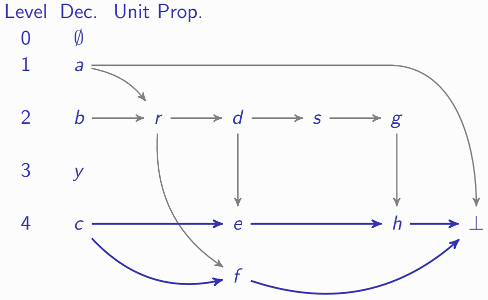
Quiz – conflict clause minimization (cont.)



Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Quiz – conflict clause minimization (cont.)

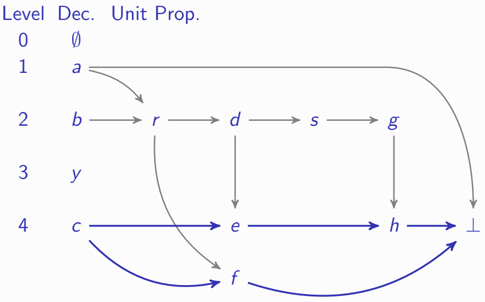


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action

Quiz – conflict clause minimization (cont.)

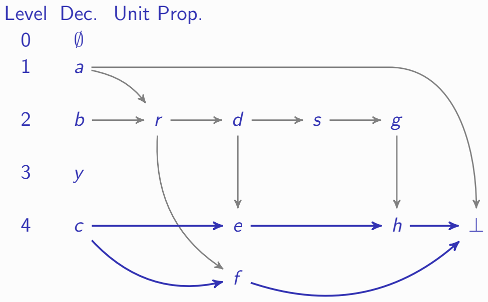


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
r	r	$\{a, c\}$	$\emptyset$	$[a, b]$	–
r	a	$\{a, c\}$	$\emptyset$	$[b]$	a marked
r	b	$\{a, c\}$	$\{b\}$	$[]$	b decision & unmarked
r	–	$\{a, c\}$	$\{b\}$	$[]$	unmarked vars; $\therefore$ keep r

Quiz – conflict clause minimization (cont.)



Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

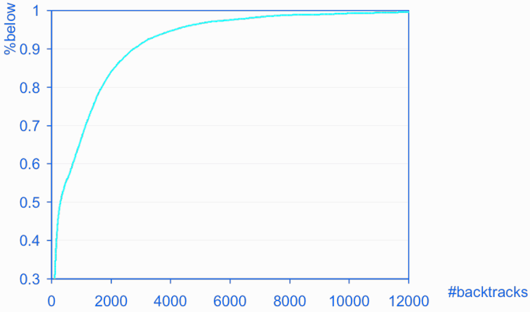
Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
r	r	$\{a, c\}$	$\emptyset$	$[a, b]$	–
r	a	$\{a, c\}$	$\emptyset$	$[b]$	a marked
r	b	$\{a, c\}$	$\{b\}$	$[]$	b decision & unmarked
r	–	$\{a, c\}$	$\{b\}$	$[]$	unmarked vars; $\therefore$ keep r
a, c	–	–	$\emptyset$	$[]$	a, c decision variables; keep both

Outline

Branch randomization

- Heavy-tail behavior: [GSC97]

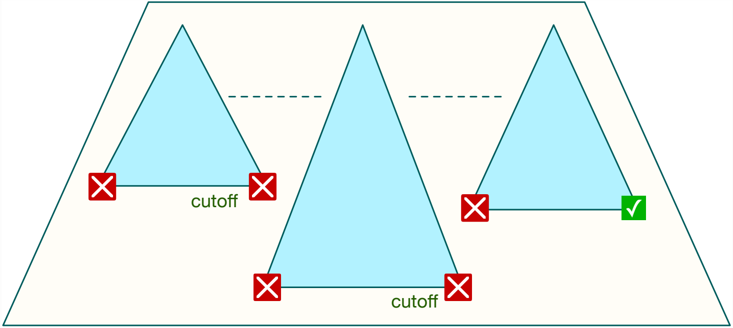


  - 10000 runs, branching randomization on satisfiable industrial instance

$\therefore$ use rapid randomized restarts (search restarts)

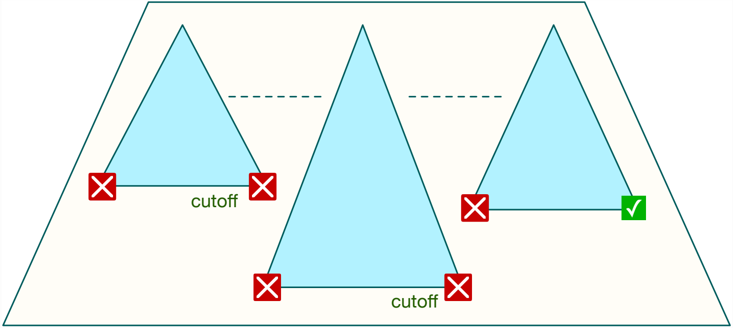
Search restarts

- Restart search after a number of conflicts



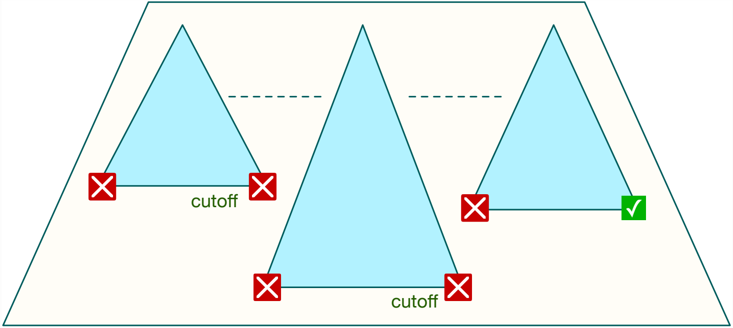
Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist



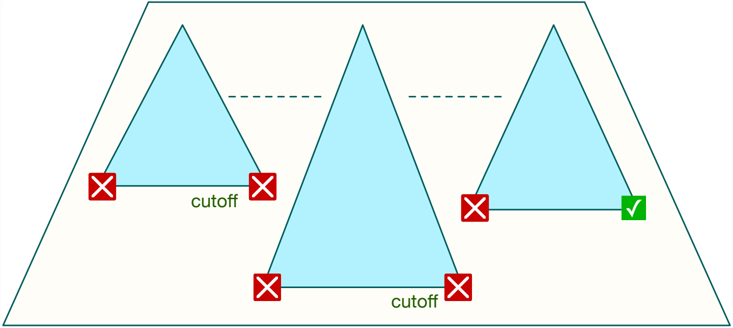
Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist
  - Effective for SAT & UNSAT formulas. **Why?**



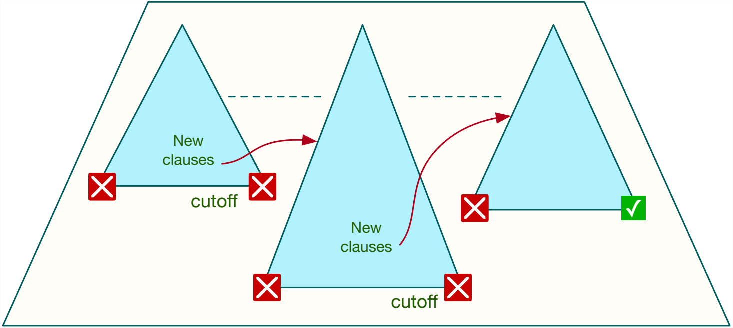
Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist
  - Effective for SAT & UNSAT formulas. **Why?**
    - Proof complexity arguments



Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist
  - Effective for SAT & UNSAT formulas. **Why?**
    - Proof complexity arguments
  - Clause learning (very) effective in between restarts



Outline

Data structures basics

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  - Why?

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  - Watched literals are one example of lazy data structures
    - But there are others [ZS00]

Watched literals

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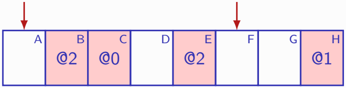
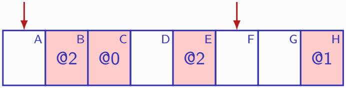


Watch 2 unassigned literals in each clause

Watched literals



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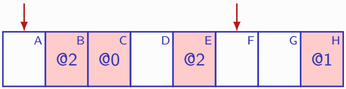

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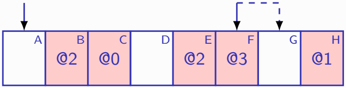

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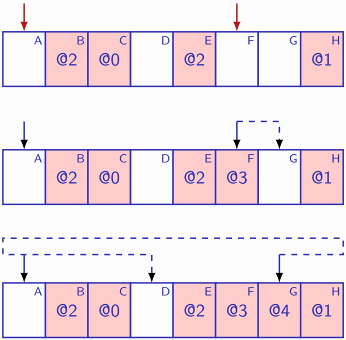

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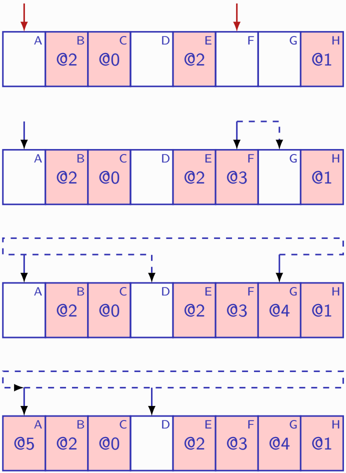

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Literal D assigned value 1; clause becomes satisfied

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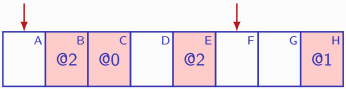

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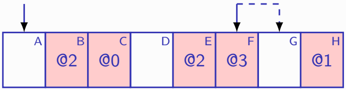

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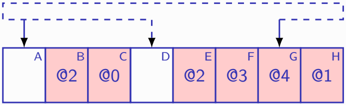

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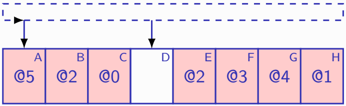

At DLevel 5: clause is unit
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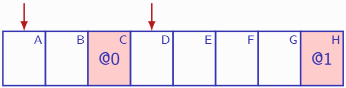
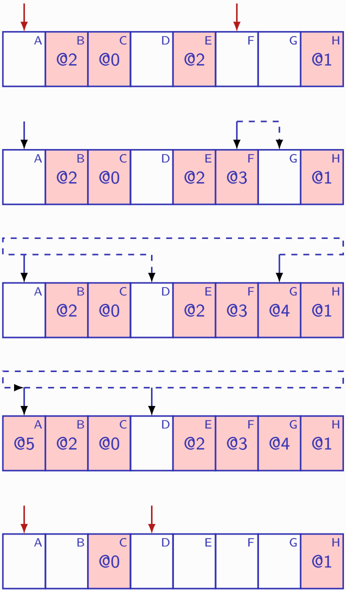


After backtracking to DLevel 1
Watched literals untouched

Watched literals – different implementations exist!



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Additional key techniques

- Conflict-driven branching [MMZ+01]
  - Use conflict to bias variables to branch on, associate score with each variable
  - Prefer recent bias by regularly decreasing variable scores
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- Other effective techniques:
  - Phase saving [PD07]
  - Novel restart strategies [Hua07, BF15, LOM+18]
  - Preprocessing/inprocessing [JHB12, HJL+15]
  - Clause minimization: LBD-based and UP-based [AS09, LLX+17]

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Why CDCL works – a practitioner's view

- GRASP-like clause learning extensively inspired in circuit reasoners
  - UIPs mimic unique sensitization points (USPs), from testing
  - Analysis of conflicts organized by decision levels
    - In circuits, branching is (mostly) on the inputs, e.g. PODEM, FAN, etc.
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- There are also proof complexity arguments [BKS04, PD09, PD11]

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  - **Any** learned clause contains explanation given working assumptions (more next)

An example

$$\mathcal{B} = \{(\bar{a} \vee b), (\bar{a} \vee c)\}$$
$$\mathcal{S} = \{(a \vee \overline{s_1}), (\bar{b} \vee \bar{c} \vee \overline{s_2}), (a \vee \bar{c} \vee \overline{s_3}), (a \vee \bar{b} \vee \overline{s_4})\}$$

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- Soft clauses $\mathcal{S}$: add indicator variables $\{s_1, s_2, s_3, s_4\}$
- E.g. given assumptions $\{s_1 = 1, s_2 = 0, s_3 = 0, s_4 = 1\}$, SAT solver handles formula:

$$\mathcal{F} = \{(\bar{a} \vee b), (\bar{a} \vee c), (a), (a \vee \bar{b})\}$$

which is satisfiable

Quiz – what happens in this case?

$$\mathcal{B} = \{(\bar{a} \vee b), (\bar{a} \vee c)\}$$
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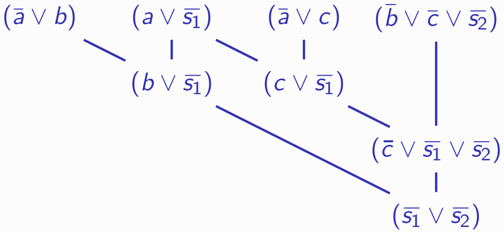

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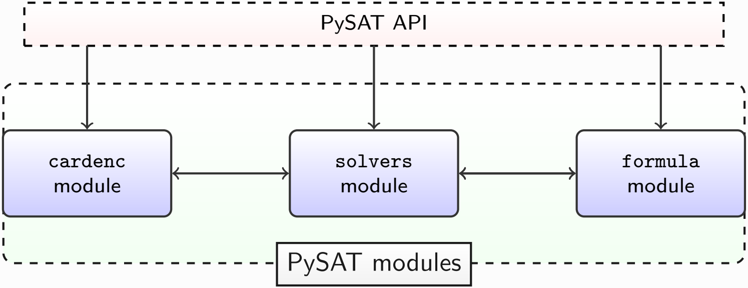
$$\begin{tikzcd}[arrows=dash]
(\bar{a} \vee b) \arrow[dr] & (a \vee \overline{s_1}) \arrow[d] \arrow[dr] & (\bar{a} \vee c) \arrow[d] & (\bar{b} \vee \bar{c} \vee \overline{s_2}) \arrow[dd] \\
& (b \vee \overline{s_1}) \arrow[ddrr] & (c \vee \overline{s_1}) \arrow[dr] & \\
& & & (\bar{c} \vee \overline{s_1} \vee \overline{s_2}) \arrow[d] \\
& & & (\overline{s_1} \vee \overline{s_2})
\end{tikzcd}$$

- Unsatisfiable core: 1st and 2nd clauses of $\mathcal{S}$, given $\mathcal{B}$

Outline

Overview of PySAT

[IMM18]



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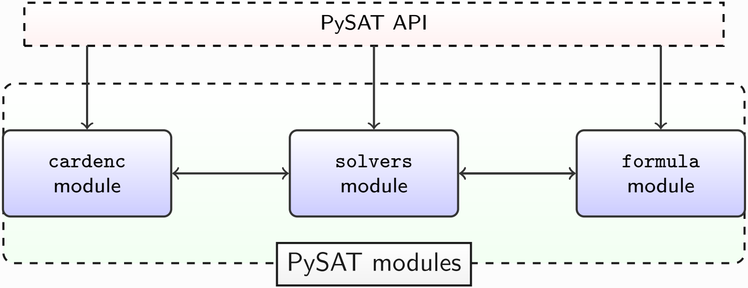

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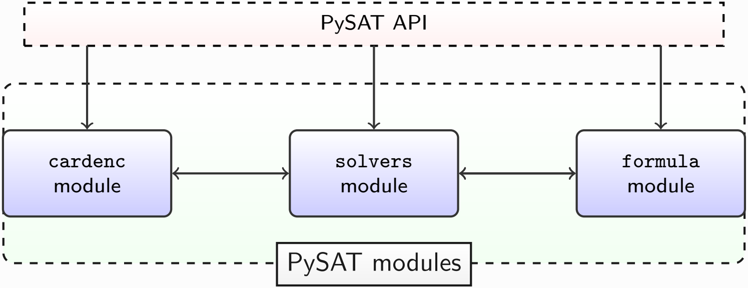

- Open source, available on github
- Comprehensive list of SAT solvers
- Comprehensive list of cardinality encodings
- Fairly comprehensive documentation
- Several use cases

Available solvers

Solver	Version
Glucose	3.0
Glucose	4.1
Lingeling	bbc-9230380-160707
Minicard	1.2
Minisat	2.2 release
Minisat	GitHub version

- Solvers can either be used incrementally or non-incrementally
- Tools can use multiple solvers, e.g. for hitting set dualization or CEGAR-based QBF solving
- **URL**: `https://pysathq.github.io/docs/html/api/solvers.html`

Formula manipulation

Features
CNF & Weighted CNF (WCNF)
Read formulas from file/string
Write formulas to file
Append clauses to formula
Negate CNF formulas
Translate between CNF and WCNF
ID manager

- **URL**: https://pysathq.github.io/docs/html/api/formula.html

Available cardinality encodings

Name	Type
pairwise	AtMost1
bitwise	AtMost1
ladder	AtMost1
sequential counter	AtMost$k$
sorting network	AtMost$k$
cardinality network	AtMost$k$
totalizer	AtMost$k$
mtotalizer	AtMost$k$
kmtotalizer	AtMost$k$

- Also AtLeast$K$ and Equals$K$ constraints
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Available cardinality encodings – more later

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Installation & info

- Installation:
  `$ [sudo] pip2|pip3 install python-sat`

- Website: https://pysathq.github.io/

Basic interface – Python3

```
>>> from pysat.card import *
>>> am1 = CardEnc.atmost(lits=[1, -2, 3], encoding=EncType.pairwise)
>>> print(am1.clauses)
[[-1, 2], [-1, -3], [2, -3]]
>>>
>>> from pysat.solvers import Solver
>>> with Solver(name='m22', bootstrap_with=am1.clauses) as s:
...     if s.solve(assumptions=[1, 2, 3]) == False:
...         print(s.get_core())
[3, 1]
```

Part 2

Problem Modeling for SAT

Quiz – solving Sudoku (first attempt)

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

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4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

- How to solve Sudoku with constraints / SAT?

A solution in Prolog CLPFD

```
:- use_module(library(clpfd)).

sudoku(Rows) :-
    length(Rows, 9),
    maplist(same_length(Rows), Rows),
    append(Rows, Vs),
    Vs ins 1..9,
    maplist(all_distinct, Rows),
    transpose(Rows, Columns),
    maplist(all_distinct, Columns),
    Rows = [As,Bs,Cs,Ds,Es,Fs,Gs,Hs,Is],
    blocks(As, Bs, Cs),
    blocks(Ds, Es, Fs),
    blocks(Gs, Hs, Is).

blocks([], [], []).
blocks([N1,N2,N3|Ns1], [N4,N5,N6|Ns2], [N7,N8,N9|Ns3]) :-
    all_distinct([N1,N2,N3,N4,N5,N6,N7,N8,N9]),
    blocks(Ns1, Ns2, Ns3).
```

A solution with Minizinc

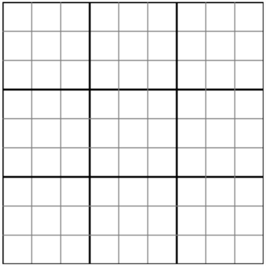
```
int: S;
int: N = S * S;
array[1..N,1..N] of var 1..N: puzzle;
include "alldifferent.mzn";

% All cells in a row, in a column, and in a subsquare are
  different.
constraint
    forall(i in 1..N)( alldifferent(j in 1..N)( puzzle[i,j] )) /\
    forall(j in 1..N)( alldifferent(i in 1..N)( puzzle[i,j] )) /\
    forall(i,j in 1..S)
        ( alldifferent(p,q in 1..S)( puzzle[S*(i-1)+p,
          S*(j-1)+q] ));

    solve satisfy;

    output [ "sudoku:\n" ] ++
    [ show(puzzle[i,j]) ++
        if j = N then
            if i mod S = 0 /\ i < N then "\n\n" else "\n" endif
        else
            if j mod S = 0 then "  " else " " endif
        endif
        | i,j in 1..N ];
```

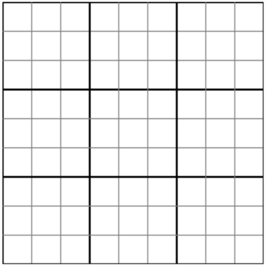
Solving Sudoku – with constraints



- Modeling the problem with integer variables:
  - Rows: $i = 1, \ldots, 9$
  - Columns: $j = 1, \ldots, 9$
  - Variables: $v_{i,j} \in \{1, 2, \ldots, 9\}$, $i, j \in \{1, \ldots, 9\}$

- Constraints:

Solving Sudoku – with constraints



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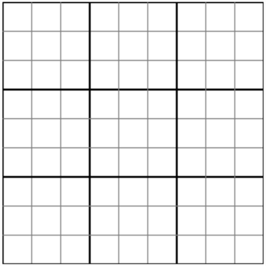
  - Each value used exactly once in each row:
    - For $i \in \{1, \ldots, 9\}$: $\text{alldifferent}(v_{i,1}, \ldots, v_{i,9})$

Solving Sudoku – with constraints



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  - Each value used exactly once in each column:
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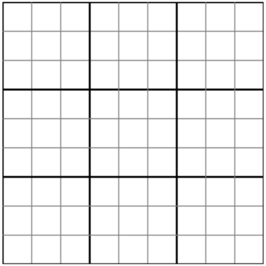
Solving Sudoku – with constraints



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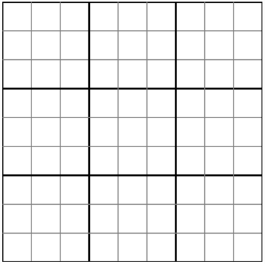
  - Each value used exactly once in each column:
    - For $j \in \{1, \ldots, 9\}$: $\text{alldifferent}(v_{1,j}, \ldots, v_{9,j})$
  - Each value used exactly once in each 3×3 sub-grid:
    - For $i, j \in \{0, 1, 2\}$:
      $\text{alldifferent}(v_{3i+1,3j+1}, v_{3i+1,3j+2}, v_{3i+1,3j+3}, v_{3i+2,3j+1}, \ldots, v_{3i+3,3j+1}, \ldots)$

Solving Sudoku – propositional logic – variables



- Modeling with propositional variables:
  - Rows: $i = 1, \ldots, 9$
  - Columns: $j = 1, \ldots, 9$
  - Variables: $v_{i,j,k} \in \{0, 1\}$, $i, j, k \in \{1, \ldots, 9\}$

Solving Sudoku – propositional logic – constraints

- Value in each cell is valid:
  - For $i, j \in \{1, \ldots, 9\}$:

$$\sum_{k=1}^{9} v_{i,j,k} = 1$$

- Each value used exactly once in each row:
  - For $i \in \{1, \ldots, 9\}$, $k \in \{1, \ldots, 9\}$:

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- Each value used exactly once in each column:
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$$\sum_{i=1}^{9} v_{i,j,k} = 1$$

- Each value used exactly once in each 3×3 sub-grid:
  - For $i, j \in \{0, 1, 2\}$, $k \in \{1, \ldots, 9\}$:

$$\sum_{r=1}^{3} \sum_{s=1}^{3} v_{3i+r,3j+s,k} = 1$$

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  - For $j \in \{1, \dots, 9\}$, $k \in \{1, \dots, 9\}$:

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$$\sum_{r=1}^{3} \sum_{s=1}^{3} v_{3i+r,3j+s,k} = 1$$

- **Q**: how to (propositionally) encode Equals1 constraints?

Constraints for fixed cells

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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6			1	9	5			
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8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Integer variables:

$$v_{1,1} = 5, v_{1,2} = 3, v_{1,5} = 7, v_{2,1} = 6, v_{2,4} = 1, v_{2,5} = 9$$
$$v_{2,6} = 5, v_{3,2} = 9, v_{3,3} = 8, v_{3,8} = 6, v_{4,1} = 8, v_{4,5} = 6, \ldots$$

Constraints for fixed cells

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Integer variables:

$$v_{1,1} = 5, v_{1,2} = 3, v_{1,5} = 7, v_{2,1} = 6, v_{2,4} = 1, v_{2,5} = 9$$
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- Propositional variables:

$$v_{1,1,5} = 1, v_{1,2,3} = 1, v_{1,5,7} = 1, v_{2,1,6} = 1, v_{2,4,1} = 1, v_{2,5,9} = 1$$
$$v_{2,6,5} = 1, v_{3,2,9} = 1, v_{3,3,8} = 1, v_{3,8,6} = 1, v_{4,1,8} = 1, v_{4,5,6} = 1, \ldots$$

Sudoku with PySAT

Demo

Outline

How to translate to CNF?

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- **Obs:** *There are no CNF formulas* [Stu13]

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  - Operator elimination; De Morgan's laws, remove double negations & apply distributivity
  - Worst-case exponential
  - Set of variables constant

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- Standard textbook solution
  - Operator elimination; De Morgan's laws, remove double negations & apply distributivity
  - Worst-case exponential
  - Set of variables constant

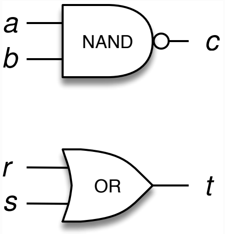
- Tseitin's translation & variants (next)
  - New variables added
  - Satisfiability is preserved
  - Linear size transformation

Representing Boolean formulas / circuits I

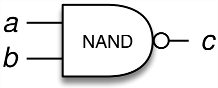
- Satisfiability problems can be defined on Boolean circuits/formulas
  - Can use any logic connective: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \ldots$
- Can represent circuits/formulas as CNF formulas [Tse68, PG86]
  - For each (simple) gate, CNF formula encodes the consistent assignments to the gate's inputs and output
    - Given $z = \mathrm{OP}(x, y)$, represent in CNF $z \leftrightarrow \mathrm{OP}(x, y)$
  - CNF formula for the circuit is the conjunction of CNF formula for each gate

$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

$$\mathcal{F}_t = (\bar{r} \vee t) \wedge (\bar{s} \vee t) \wedge (r \vee s \vee \bar{t})$$



Representing Boolean formulas / circuits II

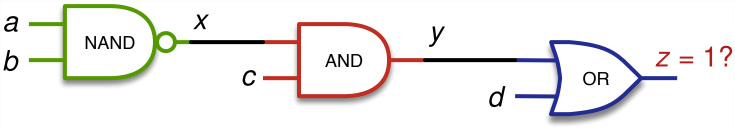


c \ ab	00	01	11	10
0	0	0	1	0
1	1	1	0	1

a	b	c	$\mathcal{F}_c$(a,b,c)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

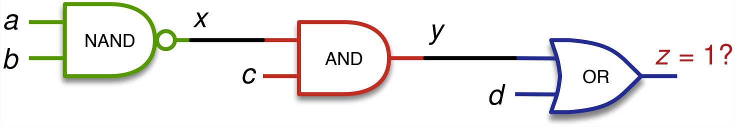
- CNF formula for the circuit is the conjunction of the CNF formula for each gate
  - Can specify objectives with additional clauses



$$\begin{aligned}\mathcal{F} \quad = \quad & (a \lor x) \land (b \lor x) \land (\bar{a} \lor \bar{b} \lor \bar{x}) \land \\ & (x \lor \bar{y}) \land (c \lor \bar{y}) \land (\bar{x} \lor \bar{c} \lor y) \land \\ & (\bar{y} \lor z) \land (\bar{d} \lor z) \land (y \lor d \lor \bar{z}) \land (z)\end{aligned}$$

Representing Boolean formulas / circuits III

- CNF formula for the circuit is the conjunction of the CNF formula for each gate
  - Can specify objectives with additional clauses



$$
\begin{aligned}
\mathcal{F} \quad = \quad & (a \vee x) \wedge (b \vee x) \wedge (\bar{a} \vee \bar{b} \vee \bar{x}) \wedge \\
& (x \vee \bar{y}) \wedge (c \vee \bar{y}) \wedge (\bar{x} \vee \bar{c} \vee y) \wedge \\
& (\bar{y} \vee z) \wedge (\bar{d} \vee z) \wedge (y \vee d \vee \bar{z}) \wedge (z)
\end{aligned}
$$

- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
  - **No** distinction between Boolean circuits and (non-clausal) formulas, besides adding new variables

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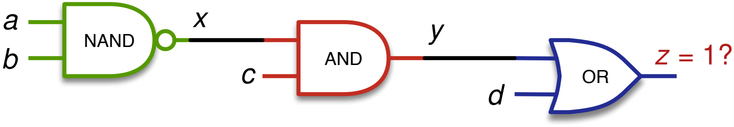

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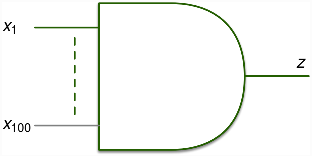
- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
  - **No** distinction between Boolean circuits and (non-clausal) formulas, besides adding new variables
- Easy to do more structures: ITEs; Adders; etc.

Quiz – how to encode a 100 input gate?



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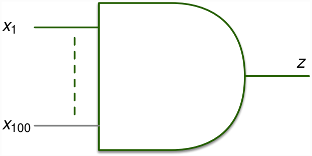

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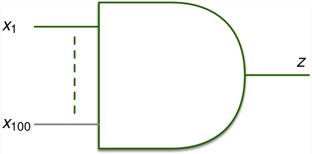

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- For any x_i, if $x_i = 0$, then $z = 0$

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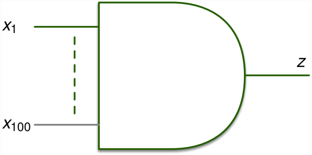

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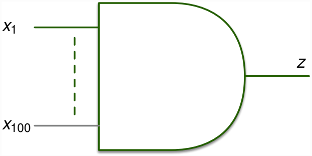

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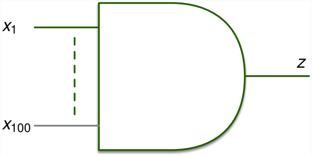

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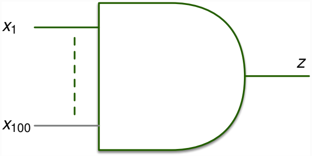

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- Resulting CNF encoding:

$$\bigwedge_{i=1}^{100} (x_i \vee \bar{z}) \wedge (\overline{x_1} \vee \cdots \vee \overline{x_{100}} \vee z)$$

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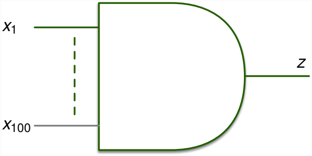

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- Similar ideas apply for other (simple) logical operators: AND, NAND, OR, NOR, etc.

Outline

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  - How to model linear cost function optimization?

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  - Hard constraints: φ
  - Soft constraints: $(\overline{x_j})$, each with cost c_j

Outline

Linear constraints

- Cardinality constraints: $\sum_{j=1}^{n} x_j \leq k$?
  - How to handle AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$?
  - General form: $\sum_{j=1}^{n} x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$

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- Pseudo-Boolean constraints: $\sum_{j=1}^{n} a_j x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$
- If variables are non-Boolean, e.g. with finite domain
  - Need to encode variables (more later)

Equals1, AtLeast1 & AtMost1 constraints

- $\sum_{j=1}^{n} x_j = 1$: encode with $(\sum_{j=1}^{n} x_j \leq 1) \wedge (\sum_{j=1}^{n} x_j \geq 1)$
- $\sum_{j=1}^{n} x_j \geq 1$: encode with $(x_1 \vee x_2 \vee \ldots \vee x_n)$
- $\sum_{j=1}^{n} x_j \leq 1$ encode with:
  - Pairwise encoding
    - Clauses: $\mathcal{O}(n^2)$; No auxiliary variables
  - Sequential counter [Sin05]
    - Clauses: $\mathcal{O}(n)$; Auxiliary variables: $\mathcal{O}(n)$
  - Bitwise encoding [FP01, Pre07]
    - Clauses: $\mathcal{O}(n \log n)$; Auxiliary variables: $\mathcal{O}(\log n)$
  - ...

Pairwise encoding

- How to (propositionally) encode AtMost1 constraint $a + b + c + d \leq 1$?

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$$
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b \rightarrow \bar{c} \wedge \bar{d} \wedge \bar{a} & \Longrightarrow & (\bar{b} \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (\bar{b} \vee \bar{a}) \\
c \rightarrow \bar{d} \wedge \bar{a} \wedge \bar{b} & \Longrightarrow & (\bar{c} \vee \bar{d}) \wedge (\bar{c} \vee \bar{a}) \wedge (\bar{c} \vee \bar{b}) \\
d \rightarrow \bar{a} \wedge \bar{b} \wedge \bar{c} & \Longrightarrow & (\bar{d} \vee \bar{a}) \wedge (\bar{d} \vee \bar{b}) \wedge (\bar{d} \vee \bar{c})
\end{array}
$$

  - Encoded as: $(\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c}) \wedge (\bar{a} \vee \bar{d}) \wedge (\bar{b} \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (\bar{c} \vee \bar{d})$

Pairwise encoding

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$$
\begin{array}{rcl}
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\end{array}
$$

  - Encoded as: $(\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c}) \wedge (\bar{a} \vee \bar{d}) \wedge (\bar{b} \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (\bar{c} \vee \bar{d})$

- With N variables, number of clauses becomes $\frac{n(n-1)}{2}$
  - But no additional variables

Sequential counter encoding

- Encode $\sum_{j=1}^{n} x_j \leq 1$ with sequential counter:

$$(\bar{x}_1 \vee s_1) \wedge (\bar{x}_n \vee \bar{s}_{n-1}) \wedge$$
$$\bigwedge_{1<i<n} ((\bar{x}_i \vee s_i) \wedge (\bar{s}_{i-1} \vee s_i) \wedge (\bar{x}_i \vee \bar{s}_{i-1}))$$

  - If some $x_j = 1$, then all s_i variables must be assigned
    - $s_i = 1$ for $i \geq j$, and so $x_i = 0$ for $i > j$
    - $s_i = 0$ for $i < j$, and so $x_i = 0$ for $i < j$
    - Thus, all other x_i variables must take value 0
  - If all $x_j = 0$, can find consistent assignment to s_i variables
  - $\mathcal{O}(n)$ clauses ; $\mathcal{O}(n)$ auxiliary variables

Bitwise encoding

- Encode $\sum_{j=1}^{n} x_j \leq 1$ with bitwise encoding:

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  - If $x_j = 1$, then $v_0 \ldots v_{r-1} = b_0 \ldots b_{r-1}$, the binary encoding of $j - 1$

    $x_j \rightarrow (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}) \Leftrightarrow (\bar{x}_j \vee (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}))$

- An example: $x_1 + x_2 + x_3 \leq 1$

	$j-1$	$v_1 v_0$
x_1	0	00
x_2	1	01
x_3	2	10

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  - Clauses $(\bar{x}_j \vee (v_i \leftrightarrow b_i)) = (\bar{x}_j \vee l_i)$, $i = 0, \ldots, r-1$, where
    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise

- An example: $x_1 + x_2 + x_3 \leq 1$

	$j-1$	$v_1 v_0$
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    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise
  - If $x_j = 1$, assignment to v_i variables must encode $j - 1$
    - For consistency, all other x variables must not take value 1
  - If all $x_j = 0$, any assignment to v_i variables is consistent
  - $\mathcal{O}(n \log n)$ clauses ; $\mathcal{O}(\log n)$ auxiliary variables
- An example: $x_1 + x_2 + x_3 \leq 1$

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$$(\bar{x}_2 \vee \bar{v}_1) \wedge (\bar{x}_2 \vee v_0)$$
$$(\bar{x}_3 \vee v_1) \wedge (\bar{x}_3 \vee \bar{v}_0)$$

General cardinality constraints

- General form: $\sum_{j=1}^{n} x_j \leq k$ (or $\sum_{j=1}^{n} x_j \geq k$)
  - Operational encoding [War98]
    - Clauses/Variables: $\mathcal{O}(n)$
    - Does not ensure arc-consistency
  - Generalized pairwise
    - Clauses: $\mathcal{O}(2^n)$; no auxiliary variables
  - Sequential counters [Sin05]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - BDDs [ES06]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - Sorting networks [Bat68, ES06]
    - Clauses/Variables: $\mathcal{O}(n \log^2 n)$
  - Cardinality Networks: [ANOR09, ANOR11]
    - Clauses/Variables: $\mathcal{O}(n \log^2 k)$
  - Pairwise Cardinality Networks: [CZ10]
  - ...

Generalized pairwise encoding

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$$
\begin{array}{lcl}
a \wedge b \rightarrow \bar{c} & \Longrightarrow & (\bar{a} \vee \bar{b} \vee \bar{c}) \\
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- In general, number of clauses is C^n_{k+1}
  - Recall: for AtMost1 (i.e. for $k = 1$), number of clauses is: $\frac{n(n-1)}{2}$

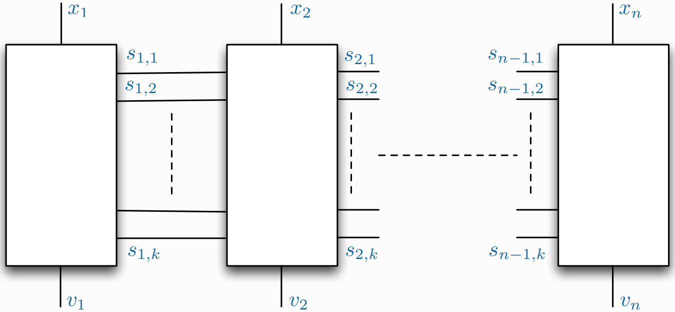
Another example

- Example: $a + b + c + d + e \leq 2$
- Encoding will contain $C_3^5 = 10$ clauses:

$$
\begin{aligned}
a \wedge b \to \bar{c} &\implies (\bar{a} \vee \bar{b} \vee \bar{c}) \\
a \wedge b \to \bar{d} &\implies (\bar{a} \vee \bar{b} \vee \bar{d}) \\
a \wedge b \to \bar{e} &\implies (\bar{a} \vee \bar{b} \vee \bar{e}) \\
a \wedge c \to \bar{d} &\implies (\bar{a} \vee \bar{c} \vee \bar{d}) \\
a \wedge c \to \bar{e} &\implies (\bar{a} \vee \bar{c} \vee \bar{e}) \\
a \wedge d \to \bar{e} &\implies (\bar{a} \vee \bar{d} \vee \bar{e}) \\
b \wedge c \to \bar{d} &\implies (\bar{b} \vee \bar{c} \vee \bar{d}) \\
b \wedge c \to \bar{e} &\implies (\bar{b} \vee \bar{c} \vee \bar{e}) \\
b \wedge d \to \bar{e} &\implies (\bar{b} \vee \bar{d} \vee \bar{e}) \\
c \wedge d \to \bar{e} &\implies (\bar{c} \vee \bar{d} \vee \bar{e})
\end{aligned}
$$

Sequential counter – revisited I

- Encode $\sum_{j=1}^{n} x_j \leq k$ with sequential counter:



- Equations for each block $1 < i < n$, $1 < j < k$:

$s_i = \sum_{j=1}^{i} x_j$

s_i represented in unary

$$s_{i,1} = s_{i-1,1} \vee x_i$$
$$s_{i,j} = s_{i-1,j} \vee s_{i-1,j-1} \wedge x_i$$
$$v_i = (s_{i-1,k} \wedge x_i) = 0$$

Sequential counter – revisited II

- CNF formula for $\sum_{j=1}^{n} x_j \leq k$:
  - Assume: $k > 0 \wedge n > 1$
  - Indeces: $1 < i < n$, $1 < j \leq k$

$$
\begin{array}{l}
(\neg x_1 \vee x_{1,1}) \\
(\neg s_{1,j}) \\
(\neg x_i \vee s_{i,1}) \\
(\neg s_{i-1,1} \vee s_{i,1}) \\
(\neg x_i \vee \neg s_{i-1,j-1} \vee s_{i,j}) \\
(\neg s_{i-1,j} \vee s_{i,j}) \\
(\neg x_i \vee \neg s_{i-1,k}) \\
(\neg x_n \vee \neg s_{n-1,k})
\end{array}
$$

- $\mathcal{O}(n\ k)$ clauses & variables

Pseudo-Boolean constraints

- General form: $\sum_{j=1}^{n} a_j x_j \leq b$
  - Operational encoding [War98]
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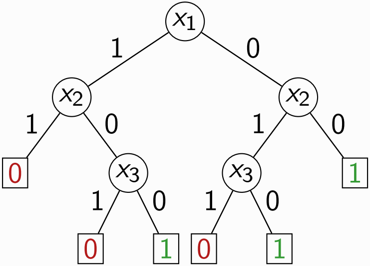
    - Let $\nu(n) = \log(n) \log(a_{max})$
    - Clauses: $\mathcal{O}(n^3 \nu(n))$; Aux variables: $\mathcal{O}(n^2 \nu(n))$
  - Improved polynomial watchdog encoding [ANO+12]
    - Clauses & aux variables: $\mathcal{O}(n^3 \log(a_{max}))$
  - ...

Encoding PB constraints with BDDs I

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Construct BDD
  - E.g. analyze variables by decreasing coefficients
- Extract ITE-based circuit from BDD



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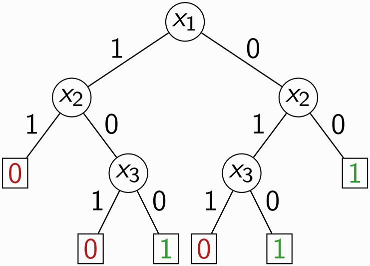
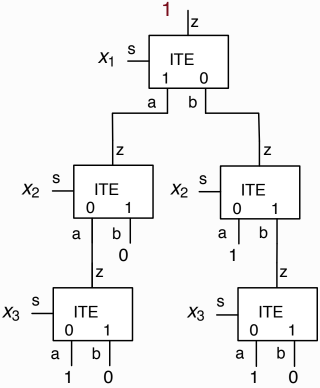
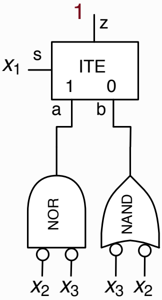




Encoding PB constraints with BDDs II

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Extract ITE-based circuit from BDD
- Simplify and create final circuit:



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$$4x_1 + 3x_2 + 2x_3 = 5$$

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  - Let $x_2 = 0$
  - Either constraint can still be satisfied, but **not** both

Outline

CSP constraints

- Many possible encodings:
  - Direct encoding [dK89, GJ96, Wal00]
  - Log encoding [Wal00]
  - Support encoding [Kas90, Gen02]
  - Log-Support encoding [Gav07]
  - Order encoding for finite linear CSPs [TTKB09]

Direct encoding for CSP w/ binary constraints

- Variable x_i with domain D_i, with $m_i = |D_i|$
- Constraints are relations over domains of variables
  - For a constraint over $x_1, \ldots, x_k$, define relation $R \subseteq D_1 \times \cdots \times D_k$
  - Need to encode elements not in the relation
  - For a binary relation, use set of binary clauses, one for each element not in R
- Represent values of x_i with Boolean variables $x_{i,1}, \ldots, x_{i,m_i}$
- Require $\sum_{k=1}^{m_i} x_{i,k} = 1$
  - Suffices to require $\sum_{k=1}^{m_i} x_{i,k} \geq 1$ [Wal00]
- If the pair of assignments $x_i = v_i \wedge x_j = v_j$ is not allowed, add binary clause $(\bar{x}_{i,v_i} \vee \bar{x}_{j,v_j})$

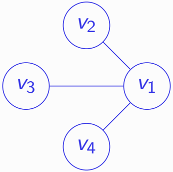
Additional topics

- Encoding problems to SAT is ubiquitous:
  - Many more encodings of finite domain CSP into SAT
  - Encodings of Answer Set Programming (ASP) into SAT
  - Eager SMT solving
  - Theorem provers iteratively encode problems into SAT
  - Model finders interatively encode problems into SAT
  - ...

Outline

Minimum vertex cover

- The problem:
  - Graph $G = (V, E)$
  - Vertex cover $U \subseteq V$
    - For each $(v_i, v_j) \in E$, either $v_i \in U$ or $v_j \in U$
  - Minimum vertex cover: vertex cover U of minimum size



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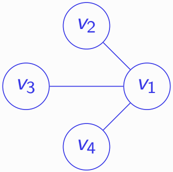

Vertex cover: $\{v_2, v_3, v_4\}$

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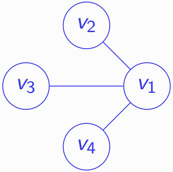

Vertex cover: $\{v_2, v_3, v_4\}$

Min vertex cover: $\{v_1\}$

Minimum vertex cover

- Modeling with Pseudo-Boolean Optimization (PBO):
  - Variables: x_i for each $v_i \in V$, with $x_i = 1$ iff $v_i \in U$
  - Clauses: $(x_i \vee x_j)$ for each $(v_i, v_j) \in E$
  - Objective function: minimize number of true x_i variables
    - I.e. minimize vertices included in U

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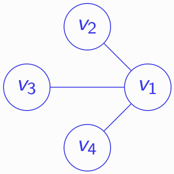

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\text{minimize} & x_1 + x_2 + x_3 + x_4 \\
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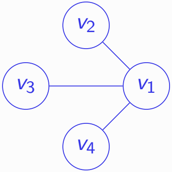

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 + x_3 + x_4 \\ \text{subject to} & (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_1 \vee x_4) \end{array}$$

- Alternative propositional encoding:

$$\begin{array}{lcl} \varphi_S & = & \{(\neg x_1), (\neg x_2), (\neg x_3), (\neg x_4)\} \\ \varphi_H & = & \{(x_1 \vee x_2), (x_1 \vee x_3), (x_1 \vee x_4)\} \end{array}$$

Graph coloring

- Given undirected graph $G = (V, E)$ and k colors:
  - Can we assign colors to vertices of G s.t. any pair of adjacent vertices are assigned different colors?

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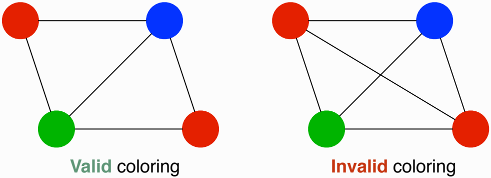

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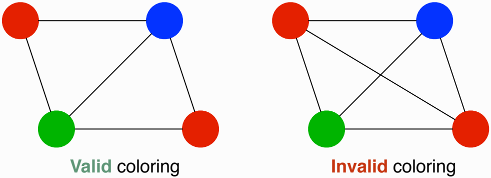

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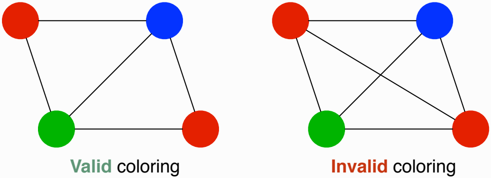

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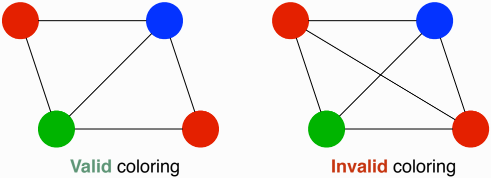

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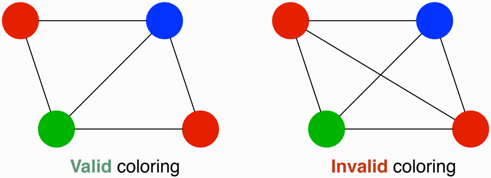

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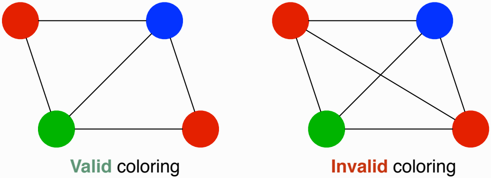

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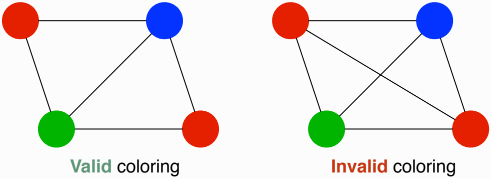

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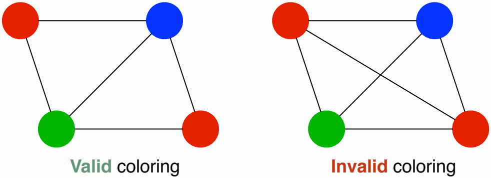

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- How to model vertices get some color?
  - $\sum_{j \in \{1,\ldots,k\}} x_{i,j} = 1$, **for** $v_i \in V$
  - **Note: it suffices to use** $\left(\bigvee_{j \in \{1,\ldots,k\}} x_{i,j}\right)$

The N-Queens problem I

- The N-Queens Problem:
  Place N queens on a $N \times N$ board, such that no two queens attack each other

- Example for a 5×5 board:

Q				
			Q	
	Q			
				Q
		Q		

The N-Queens problem II

- x_{ij}: 1 if queen placed in position (i, j); 0 otherwise
- Each row must have exactly one queen:

$$1 \leq i \leq N, \qquad \sum_{j=1}^{N} x_{ij} = 1$$

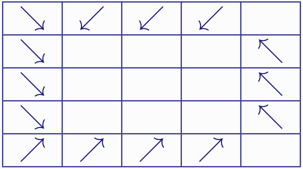
- Each column must have exactly one queen:

$$1 \leq j \leq N, \qquad \sum_{i=1}^{N} x_{ij} = 1$$

- Also, need to define constraints on diagonals...

The N-Queens problem III

- Each diagonal can have at most one queen:



$$i = 1, \quad 2 \le j < N, \quad \sum_{k=0}^{j-1} x_{i+k\, j-k} \le 1$$

$$i = N, \quad 1 \le j < N, \quad \sum_{k=0}^{N-j} x_{i-k\, j+k} \le 1$$

$$j = 1, \quad 1 \le i < N, \quad \sum_{k=0}^{N-i} x_{i+k\, j+k} \le 1$$

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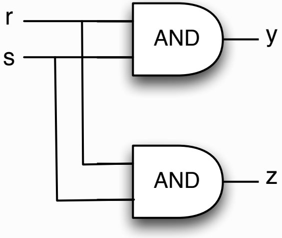
Design debugging

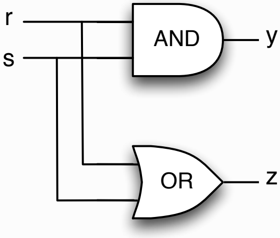
[SMV+07]

Correct circuit



Input stimuli: $\langle r, s \rangle = \langle 0, 1 \rangle$
Valid output: $\langle y, z \rangle = \langle 0, 0 \rangle$

Faulty circuit



Input stimuli: $\langle r, s \rangle = \langle 0, 1 \rangle$
Invalid output: $\langle y, z \rangle = \langle 0, 0 \rangle$

- The model:
  - Hard clauses: Input and output values
  - Soft clauses: CNF representation of circuit
- The problem:
  - Maximize number of satisfied clauses (i.e. circuit gates)

Software package upgrades

[MBC+06, TSJL07, AL08, ALS09, ABL+10b]

- Universe of software packages: $\{p_1, \ldots, p_n\}$
- Associate x_i with p_i: $x_i = 1$ iff p_i is installed
- Constraints associated with package p_i: (p_i, D_i, C_i)
  - D_i: dependencies (required packages) for installing p_i
  - C_i: conflicts (disallowed packages) for installing p_i
- Example problem: Maximum Installability
  - Maximum number of packages that can be installed
  - Package constraints represent hard clauses
  - Soft clauses: (x_i)

Package constraints:

$(p_1, \{p_2 \vee p_3\}, \{p_4\})$
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\end{array}
$$

MaxSAT formulation:

$$
\begin{array}{rcl}
\varphi_H & = & \{(\neg x_1 \vee x_2 \vee x_3), (\neg x_1 \vee \neg x_4), \\
& & (\neg x_2 \vee x_3), (\neg x_2 \vee \neg x_4), (\neg x_3 \vee x_2), \\
& & (\neg x_4 \vee x_2), (\neg x_4 \vee x_3)\} \\
\varphi_S & = & \{(x_1), (x_2), (x_3), (x_4)\}
\end{array}
$$

The knapsack problem

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- Propositional encoding for the knapsack problem?

- **Solution:** consider 0-1 ILP (or PBO) formulation:
  - Associate propositional variable x_i with each objet i
  - $x_i = 1$ iff object i is picked

$$
\begin{array}{ll}
\text{max} & \sum_{i=1}^{n} v_i \cdot x_i \\
\text{s.t} & \sum_{i=1}^{n} w_i \cdot x_i \leq W
\end{array}
$$

Part 3

Problem Solving with SAT Oracles

Computing a model

- **Q:** How to solve the FSAT problem?

  FSAT: Compute a model of a satisfiable CNF formula $\mathcal{F}$, using an NP oracle

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  - A possible algorithm:
    - Analyze each variable $x_i \in \{x_1, \ldots, x_n\} = \text{var}(\mathcal{F})$
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- FSAT is an example of a function problem
  - **Note:** FSAT can be solved with **one** SAT oracle call

Beyond decision problems

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Beyond decision problems

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Beyond decision problems

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# solutions	Counting Problems

… and beyond NP – decision and function problems

$$
\begin{tikzcd}
 & \vdots & \\
\Sigma_3^p & & \Pi_3^p \\
 & \Delta_3^p \arrow[ul] \arrow[ur] & \\
\Sigma_2^p \arrow[uu] \arrow[ur] & & \Pi_2^p \arrow[uu] \arrow[ul] \\
 & P^{NP} = \Delta_2^p \arrow[ul] \arrow[ur] & \\
NP = \Sigma_1^p \arrow[uu] \arrow[ur] & & \Pi_1^p = coNP \arrow[uu] \arrow[ul] \\
 & \Delta_0^p = \Sigma_0^p = P = \Pi_0^p = \Delta_1^p \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

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 & FP^{NP} = F\Delta_2^p \arrow[ul] \arrow[ur] & \\
FNP = F\Sigma_1^p \arrow[uu] \arrow[ur] & & F\Pi_1^p = coFNP \arrow[uu] \arrow[ul] \\
 & F\Delta_0^p = F\Sigma_0^p = FP = F\Pi_0^p = F\Delta_1^p \arrow[ul] \arrow[ur] &
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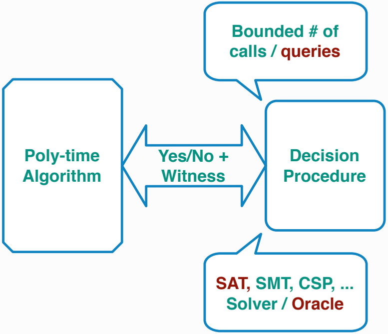
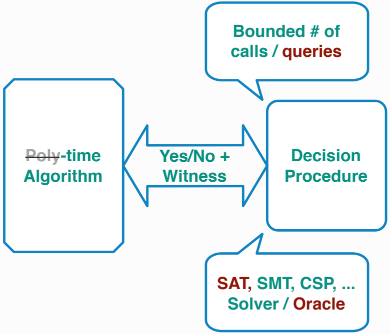
Oracle-based problem solving – ideal scenario



Oracle-based problem solving – in some settings



Many problems to solve – within $\mathsf{FP}^{\mathsf{NP}}$

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Function Problems on Propositional Formulas

MaxSAT
PBO
...
WBO
MinSAT

Minimal Models
Prime Implicants
Maximal Models
Autarkies
Backbones
Prime Implicates
...
MUSes
MCSes
MESes
Indep. Vars
MDSes
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MFSes
MSSes
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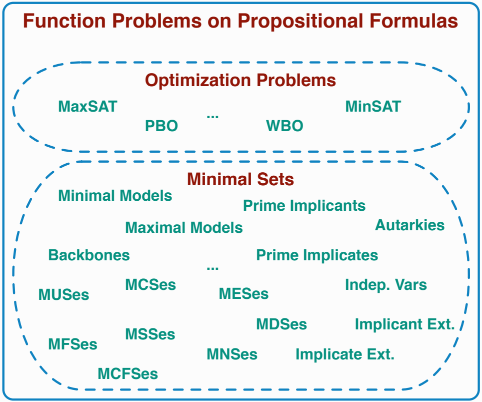
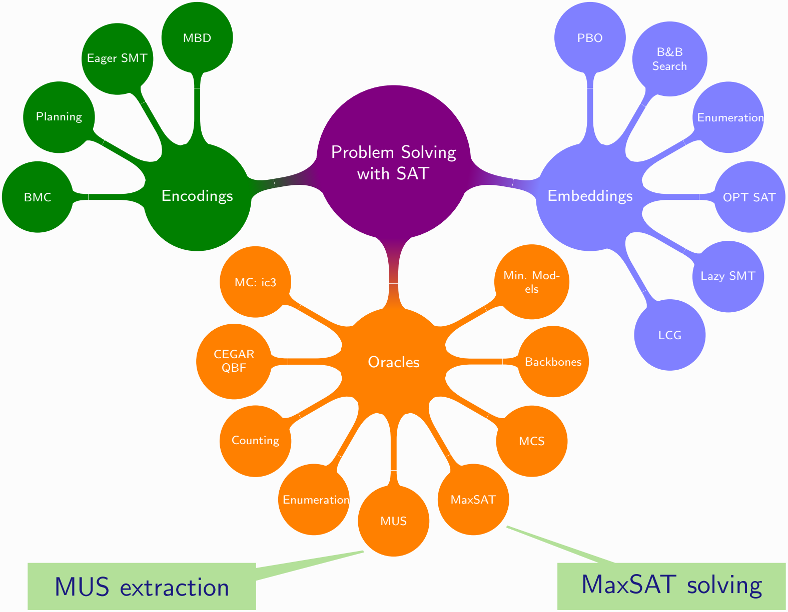


Selection of topics



Outline

Minimal Unsatisfiability

Maximum Satisfiability

Examples in PySAT

Analyzing inconsistency – timetabling

Subject	Day	Time	Room
Intro Prog	Mon	9:00-10:00	6.2.46
Intro AI	Tue	10:00-11:00	8.2.37
Databases	Tue	11:00-12:00	8.2.37
... (hundreds of consistent constraints)			
Linear Alg	Mon	9:00-10:00	6.2.46
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- How to compute these **minimal** sets?

Unsatisfiable formulas – MUSes & MCSes

- Given $\mathcal{F}$ $(\vDash \bot)$, $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \vDash \bot$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \bot$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

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- Given $\mathcal{F}$ ($\vDash \perp$), $\mathcal{C} \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus \mathcal{C} \nvDash \perp$ and $\forall_{\mathcal{C}' \subsetneq \mathcal{C}}, \mathcal{F} \setminus \mathcal{C}' \vDash \perp$. $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is MSS

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

Unsatisfiable formulas – MUSes & MCSes

- Given $\mathcal{F}$ $(\vDash \bot)$, $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \vDash \bot$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \bot$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

- Given $\mathcal{F}$ $(\vDash \bot)$, $\mathcal{C} \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus \mathcal{C} \nvDash \bot$ and $\forall_{\mathcal{C}' \subsetneq \mathcal{C}}, \mathcal{F} \setminus \mathcal{C}' \vDash \bot$. $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is MSS

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

Unsatisfiable formulas – MUSes & MCSes

- Given $\mathcal{F}$ $(\models \perp)$, $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \models \perp$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \perp$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

- Given $\mathcal{F}$ $(\models \perp)$, $\mathcal{C} \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus \mathcal{C} \nvDash \perp$ and $\forall_{\mathcal{C}' \subsetneq \mathcal{C}}, \mathcal{F} \setminus \mathcal{C}' \models \perp$. $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is MSS

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

- MUSes and MCSes are (subset-)**minimal** sets
- MUSes and minimal hitting sets of MCSes and vice-versa

[Rei87, BS05]

Unsatisfiable formulas – MUSes & MCSes

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- MUSes and MCSes are (subset-)**minimal** sets
- MUSes and minimal hitting sets of MCSes and vice-versa

[Rei87, BS05]

- How to compute MUSes & MCSes **efficiently** with SAT oracles?

Why it matters?

- Analysis of over-constrained systems
  - Model-based diagnosis [Rei87]
    - Software fault localization
    - Spreadsheet debugging
    - Debugging relational specifications (e.g. Alloy)
    - Type error debugging
    - Axiom pinpointing in description logics
    - ...
  - Model checking of software & hardware systems
  - Inconsistency measurement
  - Minimal models; MinCost SAT; ...
  - ...
- Find minimal relaxations to recover consistency
  - But also minimum relaxations to recover consistency, eg. **MaxSAT**
- **Find minimal explanations of inconsistency**
  - But also minimum explanations of inconsistency, eg. Smallest MUS

Deletion-based algorithm

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
    $\mathcal{M} \leftarrow \mathcal{F}$
    foreach $c \in \mathcal{M}$ **do**
        if $\neg\text{SAT}(\mathcal{M} \setminus \{c\})$ **then**
            $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\neg\text{SAT}(\mathcal{M} \setminus \{c\})$, then $c \notin$ MUS
    return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of oracles calls: $\mathcal{O}(m)$ [CD91, BDTW93]

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
  $\mathcal{M} \leftarrow \mathcal{F}$
  foreach $c \in \mathcal{M}$ **do**
    if $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$ **then**
      $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // Remove c from $\mathcal{M}$
  return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of oracles calls: $\mathcal{O}(m)$ [CD91, BDTW93]

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$	Outcome

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\text{SAT}(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\text{SAT}(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5
$c_4..c_7$	$c_4 c_5 c_7$	0	Keep c_6

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \lor \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \lor \neg x_4)$	(x_3)	(x_4)	$(x_5 \lor x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT$(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5
$c_4..c_7$	$c_4 c_5 c_7$	0	Keep c_6
$c_4..c_7$	$c_4..c_6$	1	Drop c_7

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg$SAT($\mathcal{M} \setminus \{c\}$)	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5
$c_4..c_7$	$c_4 c_5 c_7$	0	Keep c_6
$c_4..c_7$	$c_4..c_6$	1	Drop c_7

- MUS: $\{c_4, c_5, c_6\}$

Many MUS algorithms

- Formula $\mathcal{F}$ with m clauses k the size of largest minimal subset

Algorithm	Oracle Calls	Reference
Insertion-based	$\mathcal{O}(k\, m)$	[dSNP88, vMW08]
MCS_MUS	$\mathcal{O}(k\, m)$	[BK15]
Deletion-based	$\mathcal{O}(m)$	[CD91, BDTW93]
Linear insertion	$\mathcal{O}(m)$	[MSL11, BLM12]
Dichotomic	$\mathcal{O}(k \log(m))$	[HLSB06]
QuickXplain	$\mathcal{O}(k + k \log(\frac{m}{k}))$	[Jun04]
Progression	$\mathcal{O}(k \log(1 + \frac{m}{k}))$	[MJB13]

- **Note:** Lower bound in $\mathrm{FP}^{\mathrm{NP}}_{||}$ and upper bound in $\mathrm{FP}^{\mathrm{NP}}$ [CT95]
- Oracle calls correspond to testing unsatisfiability with SAT solver
- Practical optimizations: clause set trimming; clause set refinement; redundancy removal; (recursive) model rotation

Outline

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable

Recap MaxSAT

$\boxed{x_6 \vee x_2}$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\boxed{\neg x_4 \vee x_5}$
$\boxed{x_7 \vee x_5}$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\boxed{\neg x_1}$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$\boxed{x_7 \vee x_5}$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
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- The MaxSAT solution is one of the smallest MCSes

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest MCSes
  - **Note**: Clauses can have weights & there can be hard clauses

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest **cost** MCSes
  - **Note**: Clauses can have weights & there can be hard clauses

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
- A Minimal Correction Subset (**MCS**) is an irreducible relaxation of the formula
- The MaxSAT solution is one of the smallest **cost** MCSes
  - **Note**: Clauses can have weights & there can be hard clauses
- **Many** practical applications [SZGN17] .

MaxSAT problem(s)

		Hard Clauses? No	Hard Clauses? Yes
Weights?	No		
	Yes		

MaxSAT problem(s)

		Hard Clauses?	
		No	Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

MaxSAT problem(s)

		Hard Clauses? No	Hard Clauses? Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

- Must satisfy hard clauses, if any
- Compute set of satisfied soft clauses with maximum cost
  - Without weights, cost of each falsified soft clause is 1
- **Or**, compute set of falsified soft clauses with minimum cost (s.t. hard & remaining soft clauses are satisfied)

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		Hard Clauses?	
		No	Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

- Must satisfy hard clauses, if any
- Compute set of satisfied soft clauses with maximum cost
  - Without weights, cost of each falsified soft clause is 1
- **Or**, compute set of falsified soft clauses with minimum cost (s.t. hard & remaining soft clauses are satisfied)

- **Note**: goal is to compute **set** of satisfied (or falsified) clauses; **not** just the cost **!**

Issues with MaxSAT

- **Unit propagation is unsound for MaxSAT**

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  - Formula with all clauses soft:

$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

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  - After unit propagation:

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  - Is 2 the MaxSAT solution??

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  - **No!** Enough to either falsify (x) or (z)

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  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation

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  - **No!** Enough to either falsify (x) or (z)
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- Cannot learn clauses (using unit propagation)

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  - Formula with all clauses soft:

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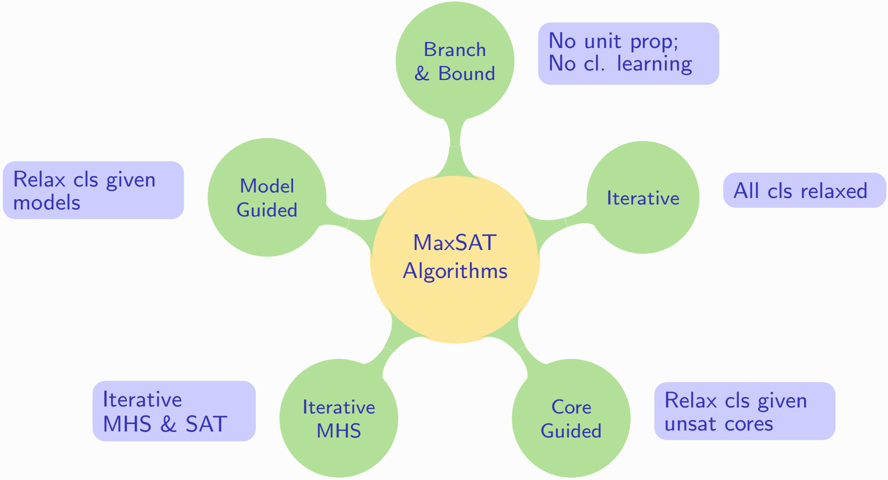
  - After unit propagation:

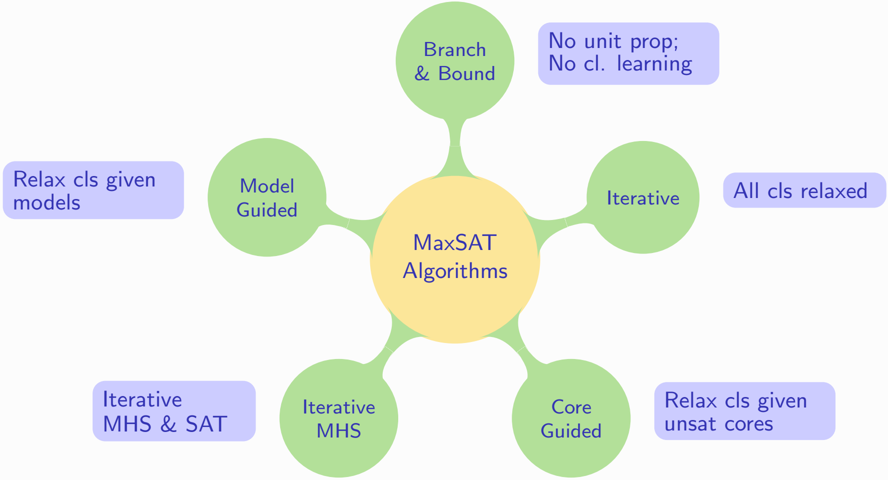
$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation
- Cannot learn clauses (using unit propagation)
- Need to solve MaxSAT using different techniques

Many MaxSAT approaches

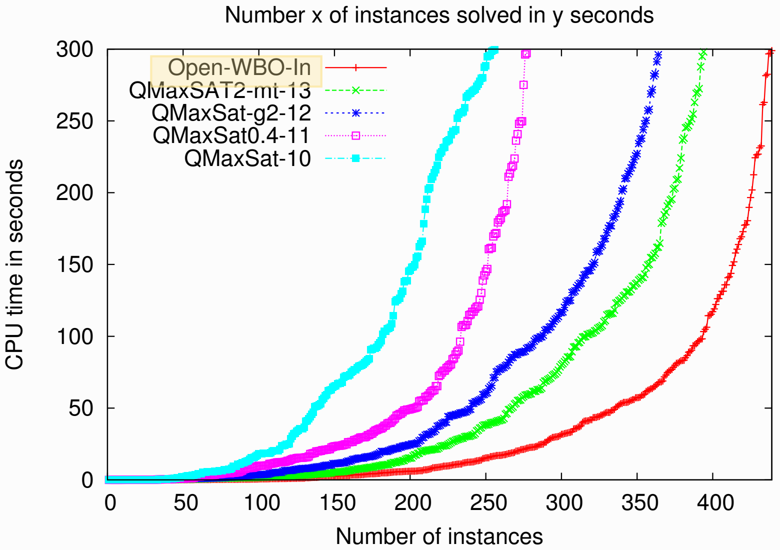


Many MaxSAT approaches



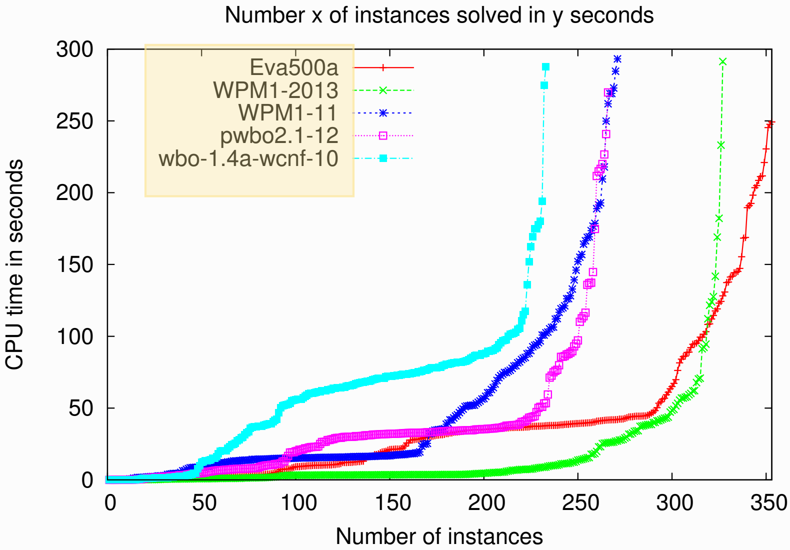
- For practical (**industrial**) instances: **core-guided** & **iterative MHS** approaches are the most effective [MaxSAT14]

Core-guided solver performance – partial



Source: [MaxSAT 2014 organizers]

Core-guided solver performance – weighted partial



Source: [MaxSAT 2014 organizers]

Outline

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Example CNF formula

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad$ $\neg x_6 \vee x_2 \vee r_2$ $\quad$ $\neg x_2 \vee x_1 \vee r_3$ $\quad$ $\neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad$ $x_6 \vee \neg x_8 \vee r_6$ $\quad$ $x_2 \vee x_4 \vee r_7$ $\quad$ $\neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\quad$ $\neg x_7 \vee x_5 \vee r_{10}$ $\quad$ $\neg x_5 \vee x_3 \vee r_{11}$ $\quad$ $\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 12$

Relax all clauses; Set $UB = 12 + 1$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 12$

Formula is SAT; E.g. all $x_i = 0$ and $r_1 = r_7 = r_9 = 1$ (i.e. cost $= 3$)

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad \neg x_6 \vee x_2 \vee r_2$ $\quad \neg x_2 \vee x_1 \vee r_3$ $\quad \neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad x_6 \vee \neg x_8 \vee r_6$ $\quad x_2 \vee x_4 \vee r_7$ $\quad \neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_{11}$ $\quad \neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 2$

Refine $UB = 3$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ | $\neg x_6 \vee x_2 \vee r_2$ | $\neg x_2 \vee x_1 \vee r_3$ | $\neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ | $x_6 \vee \neg x_8 \vee r_6$ | $x_2 \vee x_4 \vee r_7$ | $\neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ | $\neg x_7 \vee x_5 \vee r_{10}$ | $\neg x_5 \vee x_3 \vee r_{11}$ | $\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 2$

Formula is SAT; E.g. $x_1 = x_2 = 1$; $x_3 = ... = x_8 = 0$ and $r_4 = r_9 = 1$ (i.e. cost $= 2$)

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\qquad$ $\neg x_6 \vee x_2 \vee r_2$ $\qquad$ $\neg x_2 \vee x_1 \vee r_3$ $\qquad$ $\neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\qquad$ $x_6 \vee \neg x_8 \vee r_6$ $\qquad$ $x_2 \vee x_4 \vee r_7$ $\qquad$ $\neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\qquad$ $\neg x_7 \vee x_5 \vee r_{10}$ $\qquad$ $\neg x_5 \vee x_3 \vee r_{11}$ $\qquad$ $\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

Refine $UB = 2$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

Formula is UNSAT; terminate

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad \neg x_6 \vee x_2 \vee r_2$ $\quad \neg x_2 \vee x_1 \vee r_3$ $\quad \neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad x_6 \vee \neg x_8 \vee r_6$ $\quad x_2 \vee x_4 \vee r_7$ $\quad \neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_{11}$ $\quad \neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

MaxSAT solution is last satisfied UB: $UB = 2$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad$ $\neg x_6 \vee x_2 \vee r_2$ $\quad$ $\neg x_2 \vee x_1 \vee r_3$ $\quad$ $\neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad$ $x_6 \vee \neg x_8 \vee r_6$ $\quad$ $x_2 \vee x_4 \vee r_7$ $\quad$ $\neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\quad$ $\neg x_7 \vee x_5 \vee r_{10}$ $\quad$ $\neg x_5 \vee x_3 \vee r_{11}$ $\quad$ $\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

MaxSAT solution is last satisfied UB: $UB = 2$

AtMost$k$/PB constraints over **all** relaxation variables

All (possibly many) soft clauses relaxed

Outline

Minimal Unsatisfiability

Maximum Satisfiability
- Iterative SAT Solving
- Core-Guided Algorithms
- Minimum Hitting Sets

Examples in PySAT

MSU3 core-guided algorithm

$x_6 \lor x_2$	$\neg x_6 \lor x_2$	$\neg x_2 \lor x_1$	$\neg x_1$
$\neg x_6 \lor x_8$	$x_6 \lor \neg x_8$	$x_2 \lor x_4$	$\neg x_4 \lor x_5$
$x_7 \lor x_5$	$\neg x_7 \lor x_5$	$\neg x_5 \lor x_3$	$\neg x_3$

Example CNF formula

MSU3 core-guided algorithm

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Formula is UNSAT; $\text{OPT} \leq |\varphi| - 1$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2$ $\qquad \neg x_6 \vee x_2$ $\qquad \neg x_2 \vee x_1 \vee r_1$ $\qquad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\qquad x_6 \vee \neg x_8$ $\qquad x_2 \vee x_4 \vee r_3$ $\qquad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5$ $\qquad \neg x_7 \vee x_5$ $\qquad \neg x_5 \vee x_3 \vee r_5$ $\qquad \neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Add relaxation variables and AtMost$k$, $k = 1$, constraint

MSU3 core-guided algorithm

$x_6 \vee x_2$ $\quad \neg x_6 \vee x_2$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5$ $\quad \neg x_7 \vee x_5$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Formula is (again) UNSAT; OPT $\leq |\varphi| - 2$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Add new relaxation variables and update AtMost$k$, k=2, constraint

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Instance is now SAT

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad$ $\neg x_6 \vee x_2 \vee r_8$ $\quad$ $\neg x_2 \vee x_1 \vee r_1$ $\quad$ $\neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad$ $x_6 \vee \neg x_8$ $\quad$ $x_2 \vee x_4 \vee r_3$ $\quad$ $\neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad$ $\neg x_7 \vee x_5 \vee r_{10}$ $\quad$ $\neg x_5 \vee x_3 \vee r_5$ $\quad$ $\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Relaxed soft clauses become **hard**

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Some clauses not relaxed

Relaxed soft clauses become **hard**

Outline

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)?

MHS approach for MaxSAT

$$
\begin{array}{llll}
c_1 = x_6 \vee x_2 & c_2 = \neg x_6 \vee x_2 & c_3 = \neg x_2 \vee x_1 & c_4 = \neg x_1 \\
c_5 = \neg x_6 \vee x_8 & c_6 = x_6 \vee \neg x_8 & c_7 = x_2 \vee x_4 & c_8 = \neg x_4 \vee x_5 \\
c_9 = x_7 \vee x_5 & c_{10} = \neg x_7 \vee x_5 & c_{11} = \neg x_5 \vee x_3 & c_{12} = \neg x_3
\end{array}
$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)?

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No
- Core of $\mathcal{F}$: $\{c_9, c_{10}, c_{11}, c_{12}\}$

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
$c_9 = x_7 \vee x_5$	$c_{10} = \neg x_7 \vee x_5$	$c_{11} = \neg x_5 \vee x_3$	$c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No
- Core of $\mathcal{F}$: $\{c_9, c_{10}, c_{11}, c_{12}\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$$
\begin{array}{llll}
c_1 = x_6 \vee x_2 & c_2 = \neg x_6 \vee x_2 & c_3 = \neg x_2 \vee x_1 & c_4 = \neg x_1 \\
c_5 = \neg x_6 \vee x_8 & c_6 = x_6 \vee \neg x_8 & c_7 = x_2 \vee x_4 & c_8 = \neg x_4 \vee x_5 \\
c_9 = x_7 \vee x_5 & c_{10} = \neg x_7 \vee x_5 & c_{11} = \neg x_5 \vee x_3 & c_{12} = \neg x_3
\end{array}
$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$\begin{array}{llll} c_1 = x_6 \vee x_2 & c_2 = \neg x_6 \vee x_2 & c_3 = \neg x_2 \vee x_1 & c_4 = \neg x_1 \\ c_5 = \neg x_6 \vee x_8 & c_6 = x_6 \vee \neg x_8 & c_7 = x_2 \vee x_4 & c_8 = \neg x_4 \vee x_5 \\ c_9 = x_7 \vee x_5 & c_{10} = \neg x_7 \vee x_5 & c_{11} = \neg x_5 \vee x_3 & c_{12} = \neg x_3 \end{array}$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)?

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No
- Core of $\mathcal{F}$: $\{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}, \{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No
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- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
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MHS approach for MaxSAT

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- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)? Yes
- Terminate & return 2

MaxSAT solving with SAT oracles – a sample

- A sample of recent algorithms:

Algorithm	# Oracle Queries	Reference
Linear search SU	Exponential***	[BP10]
Binary search	Linear*	[FM06]
FM/WMSU1/WPM1	Exponential**	[FM06, MP08, MMSP09, ABL09, ABGL12]
WPM2	Exponential**	[ABL10a, ABL13]
Bin-Core-Dis	Linear	[HMM11, MHM12]
Iterative MHS	Exponential	[DB11, DB13a, DB13b]

* $\mathcal{O}(\log m)$ queries with SAT oracle, for (partial) unweighted MaxSAT

** Weighted case; depends on computed cores

*** On # bits of problem instance (due to weights)

- But also additional recent work:
  - Progression
  - Soft cardinality constraints (OLL)
  - MaxSAT resolution
  - ...

Outline

Minimal Unsatisfiability

Maximum Satisfiability

Examples in PySAT

Example: naive (deletion) MUS extraction

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
    $\mathcal{M} \leftarrow \mathcal{F}$
    foreach $c \in \mathcal{M}$ **do**
        if $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$ **then**
            $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$, then $c \notin$ MUS
    return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of predicate tests: $\mathcal{O}(m)$ [CD91, BDTW93]

Naive MUS extraction I

```python
def main():
    cnf = CNF(from_file=argv[1])   # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__== "__main__":
  main()
```

Naive MUS extraction II

```python
def add_assumps(cnf):
    rnv = topv = cnf.nv
    assumps = []                        # list of assumptions to use
    for i in range(len(cnf.clauses)):
        topv += 1
        assumps.append(topv)            # register literal
        cnf.clauses[i].append(-topv)    # extend clause with literal
    cnf.nv = cnf.nv + len(assumps)      # update # of vars
    return rnv, assumps


def main():
    cnf = CNF(from_file=argv[1])    # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__ == "__main__":
  main()
```

Naive MUS extraction III

```
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Solver


def find_mus(assmp, oracle):
    i = 0
    while i < len(assmp):
        ts = assmp[:i] + assmp[(i+1):]
        if not oracle.solve(assumptions=ts):
            assmp = ts
        else:
            i += 1
    return assmp
```

Naive MUS extraction III

```
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Solver


def find_mus(assmp, oracle):
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    while i < len(assmp):
        ts = assmp[:i] + assmp[(i+1):]
        if not oracle.solve(assumptions=ts):
            assmp = ts
        else:
            i += 1
    return assmp
```

Demo

A less naive MUS extractor

```python
def clset_refine(assmp, oracle):
    assmp = sorted(assmp)
    while True:
        oracle.solve(assumptions=assmp)
        ts = sorted(oracle.get_core())
        if ts == assmp:
            break
        assmp = ts
    return assmp
# ...
def main():
    cnf = CNF(from_file=argv[1])   # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    assumps = clset_refine(assumps, oracle)
    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__ == "__main__":
  main()
```

Encoding sudoku

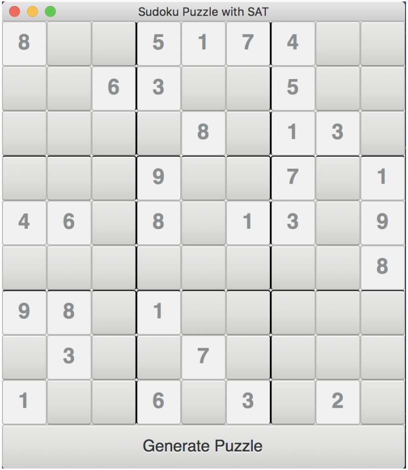
```python
class SudokuEncoding(CNF, object):
  def __init__(self):
      # initializing CNF's internal parameters
      super(SudokuEncoding, self).__init__()
      self.vpool = IDPool()
      # at least one value in each cell
      for i, j in itertools.product(range(9), range(9)):
          self.append([self.var(i, j, val) for val in range(9)])
      # at most one value in each row
      for i in range(9):
          for val in range(9):
              for j1, j2 in itertools.combinations(range(9), 2):
                  self.append([-self.var(i, j1, val), -self.var(i, j2, val)])
      # at most one value in each column
      for j in range(9):
          for val in range(9):
              for i1, i2 in itertools.combinations(range(9), 2):
                  self.append([-self.var(i1, j, val), -self.var(i2, j, val)])
      # at most one value in each square
      for val in range(9):
          for i in range(3):
              for j in range(3):
                  subgrid = itertools.product(range(3*i, 3*i+3), range(3*j, 3*j+3))
                  for c in itertools.combinations(subgrid, 2):
                      self.append([-self.var(c[0][0],c[0][1],val),
                                   -self.var(c[1][0],c[1][1],val)])

  def var(self, i, j, v):
      return self.vpool.id(tuple([i + 1, j + 1, v + 1]))

  def cell(self, var):
      return self.vpool.obj(var)
```

A prototype sudoku game

A prototype sudoku game



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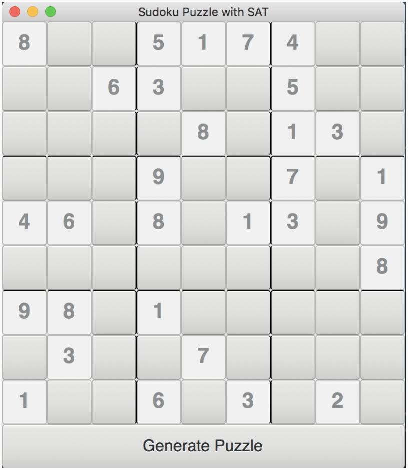


[Demo]

Part 4

Sample of Applications

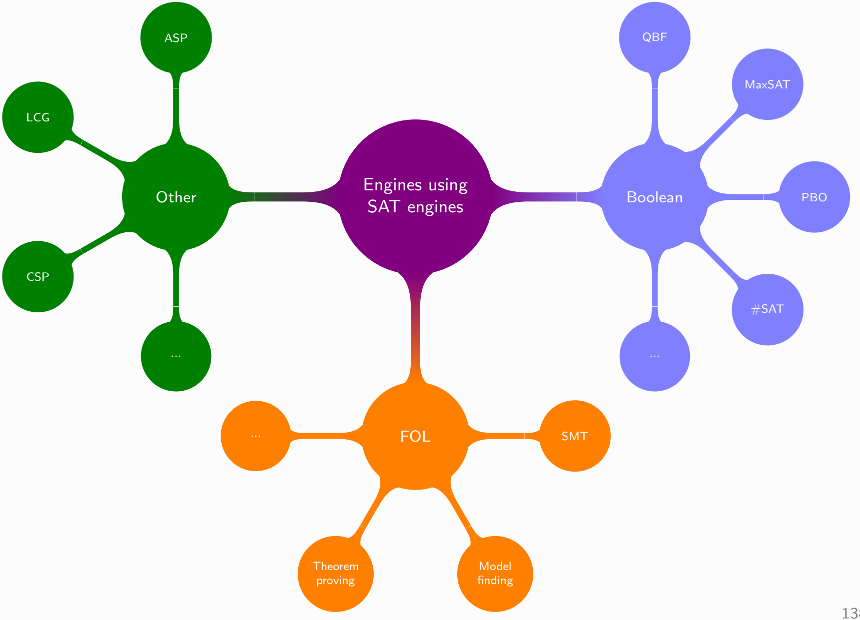
Flagship applications

- Bounded (& unbounded) model checking
- Automated planning

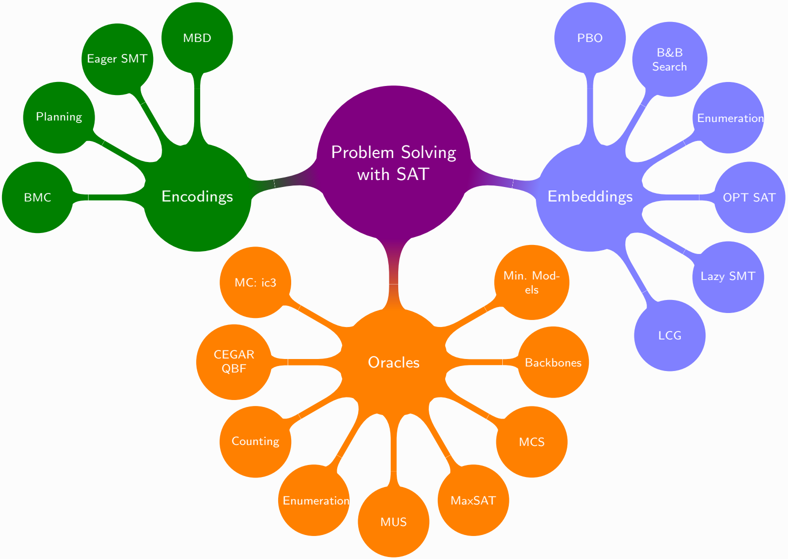
- Software model checking
- Package management
- Design debugging

- Haplotyping

CDCL SAT is **the** engines' engine



CDCL SAT is ubiquitous in problem solving



Recent applications

- Two-level logic minimization with SAT [IPM15]
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  - Computation of smallest decision trees
- Abduction-based explanations for ML models [INMS19]
  - On-demand extraction of explanations for any ML model

Smallest decision trees – encoding sizes in bytes

[NIPM18]

Model	Weather	Mouse	Cancer	Car	Income
CP'09*	27K	3.5M	92G	842M	354G

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Model	Weather	Mouse	Cancer	Car	Income
CP'09*	27K	3.5M	92G	842M	354G
IJCAI'18	190K	1.2M	5.2M	4.1M	1.2G

Abduction-based explanations

[INMS19]

- Positive:
  - General approach, applicable to **any** ML model represented as a set of constraints
  - Ability to explain predictions of NNs

- Negative:
  - NN sizes are fairly small, i.e. tens of neurons
  - Best results with ILP-based approach
    - SMT/SAT models currently ineffective
    - But, algorithms inspired SAT-based solutions

Outline

Solving MaxClique with SAT

Modeling MaxClique with SAT

- Given (undirected) graph, find largest complete subgraph
- Main constraint:

> Given $u, v \in V$:
> If $(u, v) \notin E$, then one **must not** have both u and v in the maximum-size clique

Modeling MaxClique with SAT

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> Given $u, v \in V$:
> If $(u, v) \notin E$, then one **must not** have both u and v in the maximum-size clique

- Associate Boolean x_u with $u \in V$

- Main goal:

> Assign 1 to largest set of variables that are consistent with constraint

  - E.g. use MaxSAT

An example

Construct $\mathcal{F} = \langle \mathcal{H}, \mathcal{S} \rangle$

An example

Construct $\mathcal{F} = \langle \mathcal{H}, \mathcal{S} \rangle$ s.t. $\begin{cases} \mathcal{H} & \triangleq & \{(\neg x_u \vee \neg x_v) \mid (u, v) \in E^C\} \\ \mathcal{S} & \triangleq & \{(x_u) \mid v \in V\} \end{cases}$

$$\mathcal{H} = \left\{ \begin{array}{ll} (\neg x_1 \vee \neg x_6) & (\neg x_1 \vee \neg x_7) \\ (\neg x_2 \vee \neg x_6) & (\neg x_2 \vee \neg x_7) \\ (\neg x_4 \vee \neg x_6) & (\neg x_4 \vee \neg x_7) \\ (\neg x_6 \vee \neg x_7) & \end{array} \right\}$$

$$\mathcal{S} = \left\{ \begin{array}{lll} (x_1) & (x_2) & (x_3) \\ (x_4) & (x_5) & (x_6) \\ (x_7) & & \end{array} \right\}$$

solve $\mathcal{F}$ with MaxSAT !

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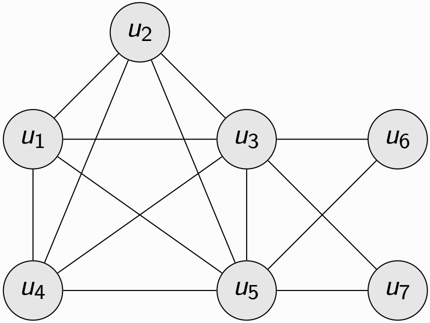

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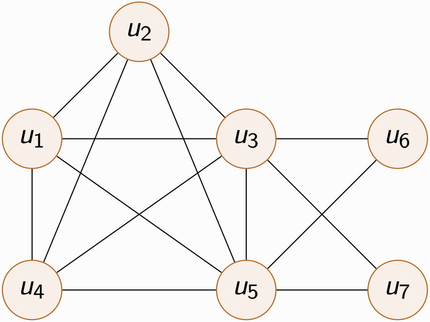

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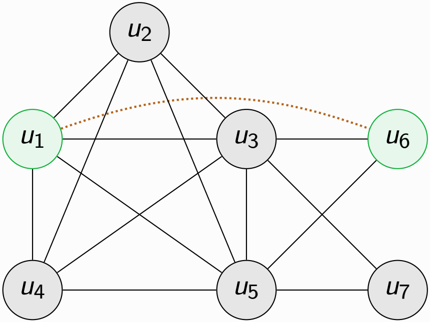

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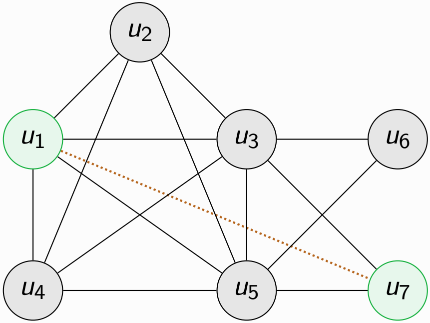

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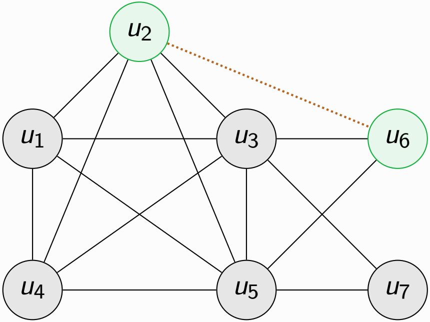

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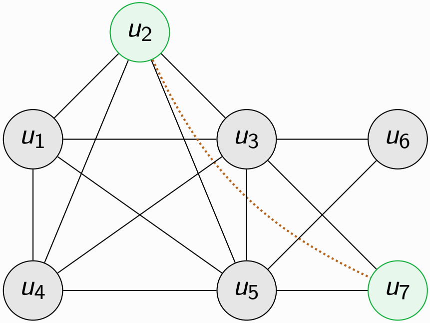

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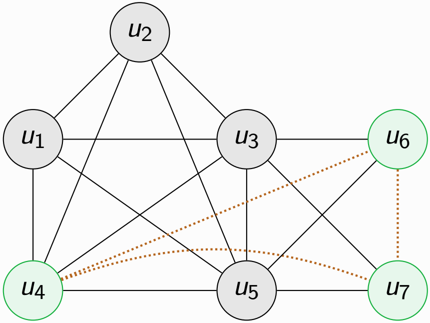
An example

Construct $\mathcal{F} = \langle \mathcal{H}, \mathcal{S} \rangle$ s.t. $\begin{cases} \mathcal{H} & \triangleq & \{(\neg x_u \lor \neg x_v) \mid (u, v) \in E^C\} \\ \mathcal{S} & \triangleq & \{(x_u) \mid v \in V\} \end{cases}$



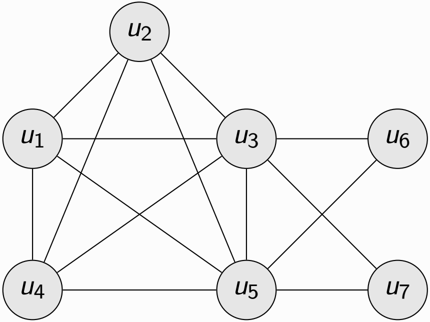
$$\mathcal{H} = \left\{ \begin{array}{l} (\neg x_1 \lor \neg x_6)\,(\neg x_1 \lor \neg x_7) \\ (\neg x_2 \lor \neg x_6)\,(\neg x_2 \lor \neg x_7) \\ (\neg x_4 \lor \neg x_6)\,(\neg x_4 \lor \neg x_7) \\ (\neg x_6 \lor \neg x_7) \end{array} \right\}$$

$$\mathcal{S} = \left\{ \begin{array}{l} (x_1)\,(x_2)\,(x_3) \\ (x_4)\,(x_5)\,(x_6) \\ (x_7) \end{array} \right\}$$

solve $\mathcal{F}$ with MaxSAT !

An example

Construct $\mathcal{F} = \langle \mathcal{H}, \mathcal{S} \rangle$ s.t. $\begin{cases} \mathcal{H} & \triangleq & \{(\neg x_u \vee \neg x_v) \mid (u, v) \in E^C\} \\ \mathcal{S} & \triangleq & \{(x_u) \mid v \in V\} \end{cases}$



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But the size of E^C can be **problematic**...

Instance	$\|V\|$	$\|E\|$	$\|E\|^C$
comm-n10000	10000	10000	49995000
ca-AstroPh	18772	396160	175807218
ca-citeseer	227322	814136	25836945367
ca-coauthors-dblp	540488	15245731	146048663585
ca-CondMat	23133	186936	267392475
ca-dblp-2010	226415	716462	25631272858
ca-dblp-2012	317082	1049868	50269606035
ca-HepPh	12008	237010	71865026
ca-HepTh	9877	51971	48730532
ca-MathSciNet	332689	820644	55340331061
ia-email-EU	32430	54397	525814268
ia-reality-call	6809	9484	23175161
ia-retweet-pol	18470	61157	170518528
ia-wiki-Talk	92117	360767	4242456136
rt-pol	18470	61157	170518528
rt_barackobama	9631	9826	46373070
soc-epinions	63947	606512	2044034866
soc-gplus	23628	39242	279113764
tech-as-caida2007	26477	53383	350475620
tech-internet-as	40164	85123	806508407
tech-pgp	10680	24340	57012200
tech-WHOIS	7476	56943	27892083
web-arabic-2005	163598	1747269	13380487332
web-baidu-baike-related	415641	3284387	86375643874
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$$|E^C| = \frac{|E|\times(|E|-1)}{2} - |E|$$

Unrealistic to model with SAT on sparse graphs

How to reduce the encoding size?

- Main hurdle:

SAT-based approaches based on $G^C = (V, E^C)$
will not scale...
And $G = (V, E)$ is **much smaller** than $G^C = (V, E^C)$

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  And $G = (V, E)$ is **much smaller** than $G^C = (V, E^C)$

- Can we model MaxClique using solely G?

Another take at solving MaxClique with SAT

- Revisit the original decision problem:

> Given $G = (V, E)$, is there a clique of size K?

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- First, one **must** pick exactly K vertices:

$$\sum_{u \in V} x_u = K$$

- And second, if a vertex $u \in V$ is picked (i.e. $x_u = 1$), **then** $K - 1$ of its neighbours **must** also be picked!

$$x_u \rightarrow \left(\sum_{v \in \mathrm{Adj}(u)} x_v = K - 1 \right)$$

Part 5

A Glimpse of the Future

What next?

- Oracle-based computing
  - Problems beyond NP: optimization, quantification, enumeration, (approximate) counting

- ...

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  - Cutting Planes (CP)
  - Extended Resolution (and equivalent)
- Verification of ML models with SAT/SMT
- Scalable explainable AI/ML
  - Deep NNs operate as black-boxes
  - Often important to provide small/intuitive explanations for predictions made
- ...

Some final notes

- SAT is a low-level, but very powerful problem solving paradigm
  - PySAT suggests a way to tackle this drawback, but there are others
- There is an ongoing revolution on problem solving with SAT oracles
- The use of SAT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.

Some final notes

- SAT is a low-level, but very powerful problem solving paradigm
  - PySAT suggests a way to tackle this drawback, but there are others
- There is an ongoing revolution on problem solving with SAT oracles
- The use of SAT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.
- Many fascinating research topics out there !
  - Connections with ML seem unavoidable

Sample of tools

- PySAT
- SAT solvers:
  - MiniSat
  - Glucose
- MaxSAT solvers:
  - RC2
  - MSCG
  - OpenWBO
  - MaxHS
- MUS extractors:
  - MUSer
- MCS extractors:
  - mcsXL
  - LBX
  - MCSls
- Many other tools available from the ReasonLab server

Questions?

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